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# COM783 – Mecânica de Sistemas Inteligentes: Revisão de Dinâmica Não-Linear e Caos

Prof. Luã G. Costa

Universidade Federal do Rio de Janeiro (COPPE/UFRJ)

Programa de Engenharia Mecânica (PEM)

# Lista 1 e Bibliografia

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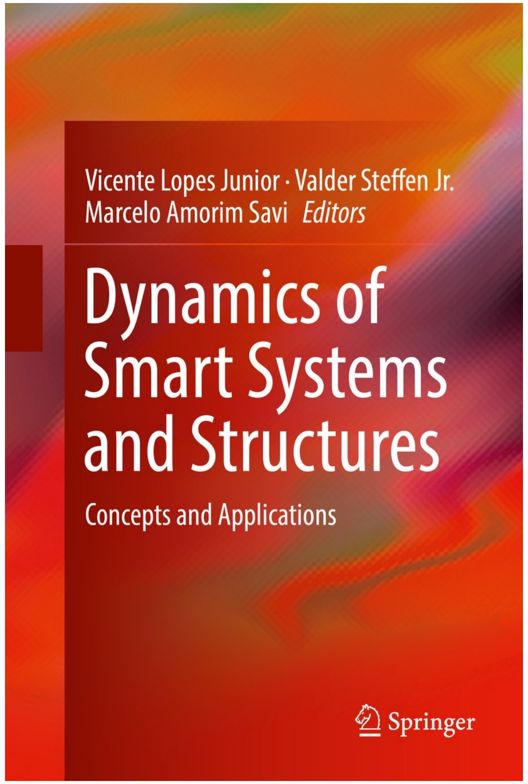
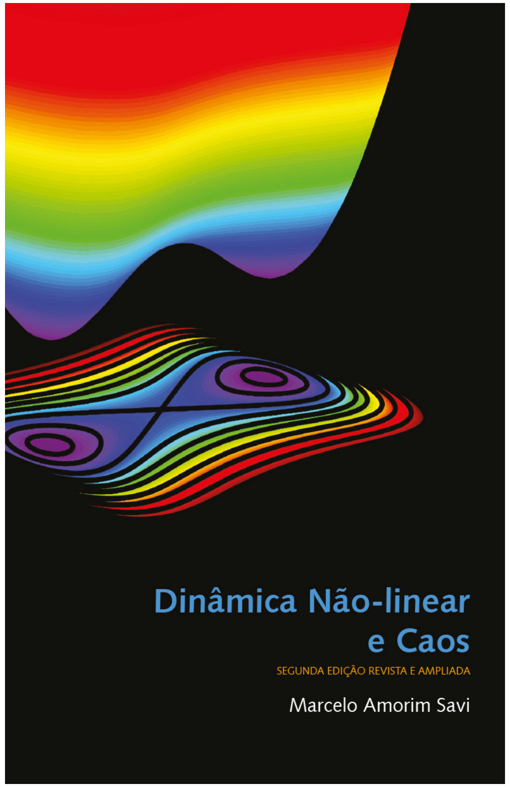
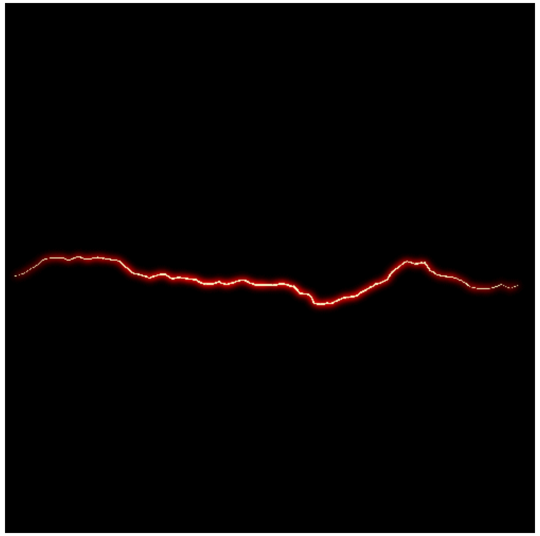
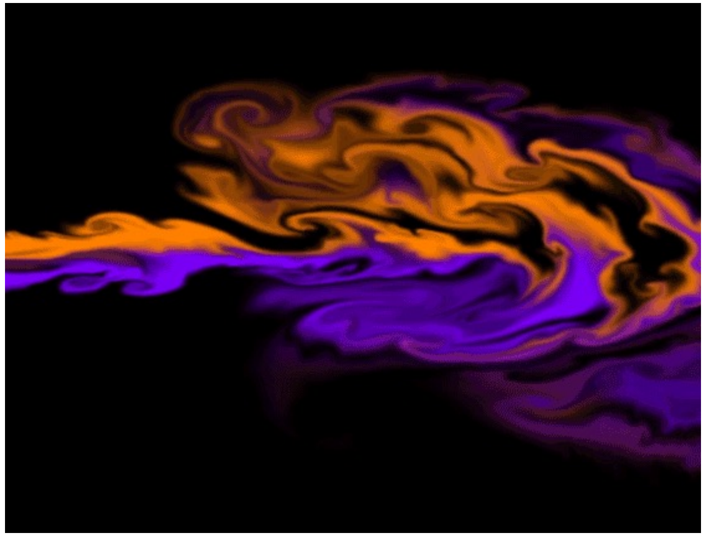
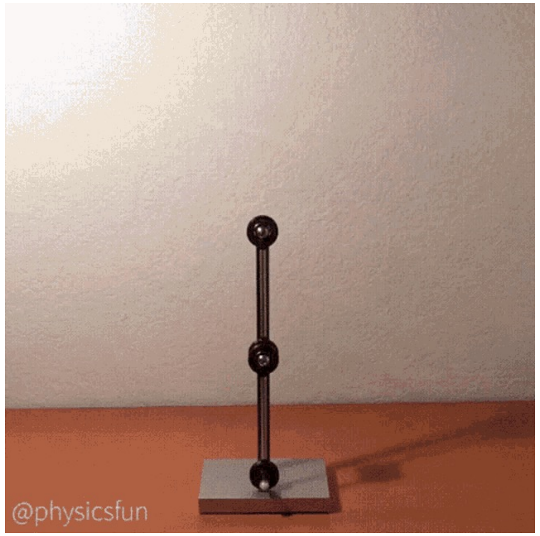
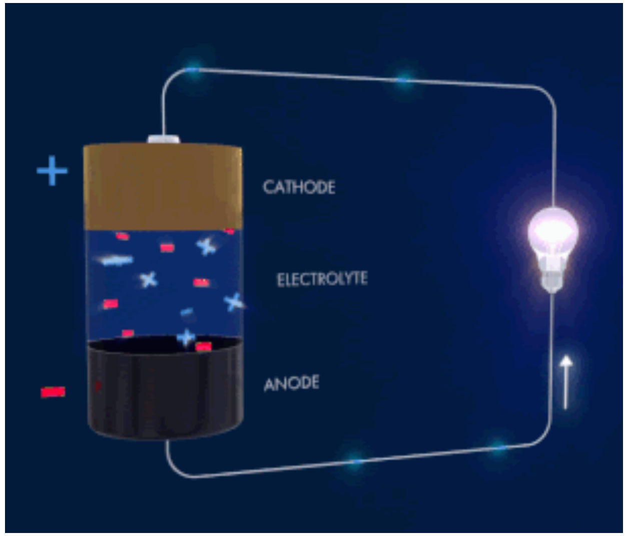
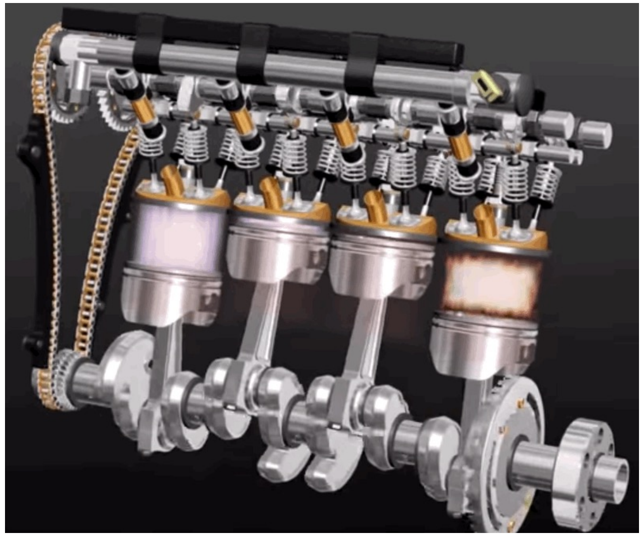


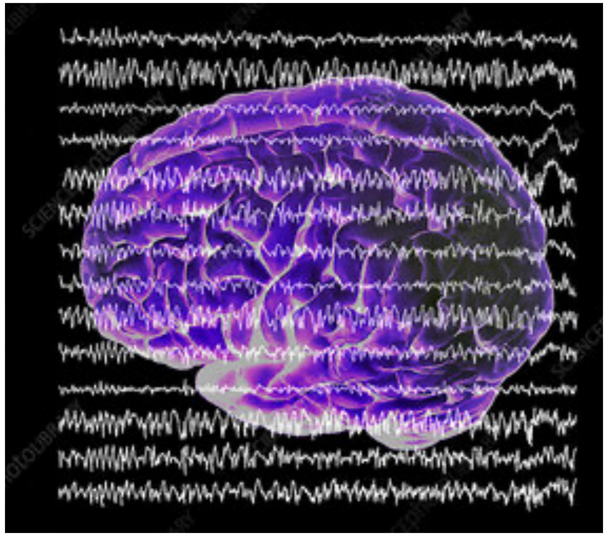
# Sistemas Dinâmicos (Engenharia)

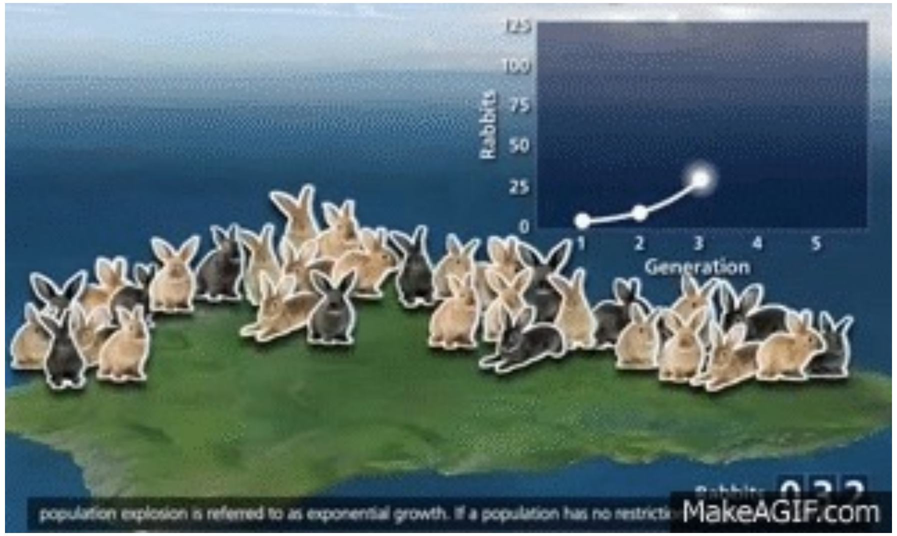





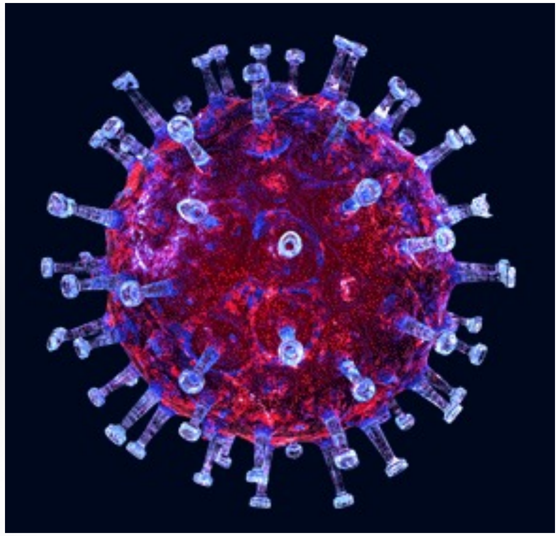


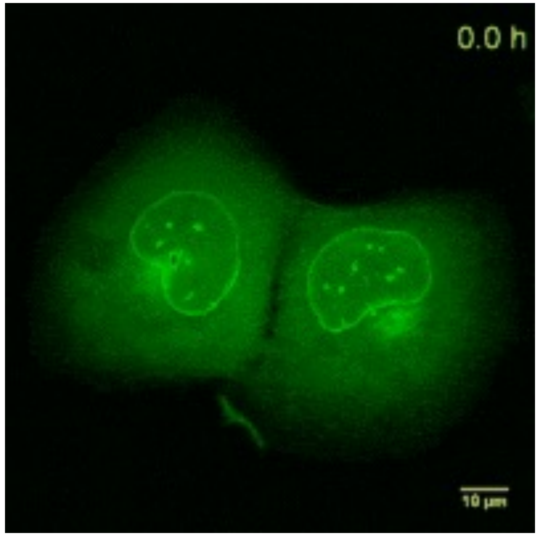



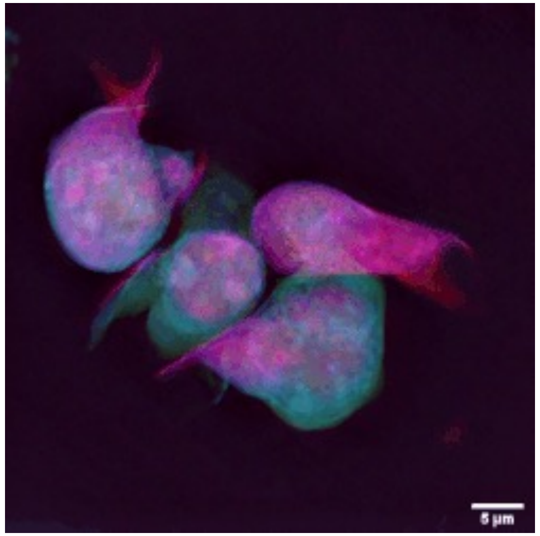
# Sistemas Dinâmicos (Biológicos)



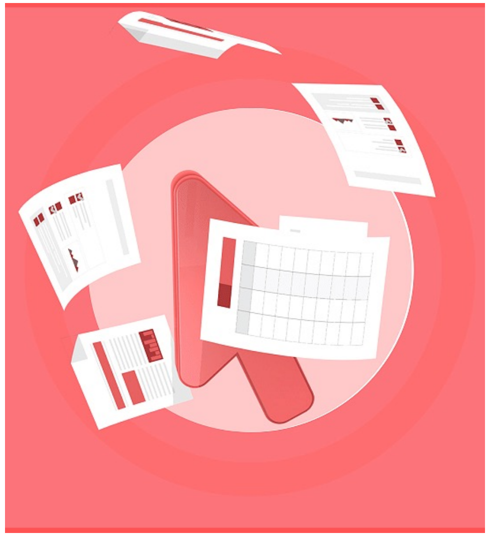




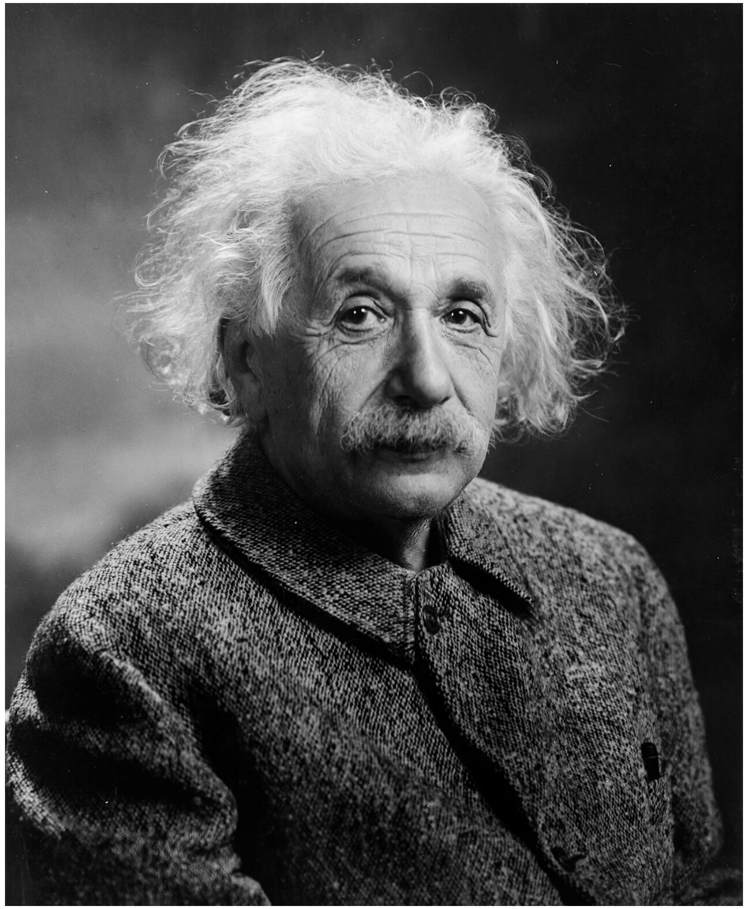



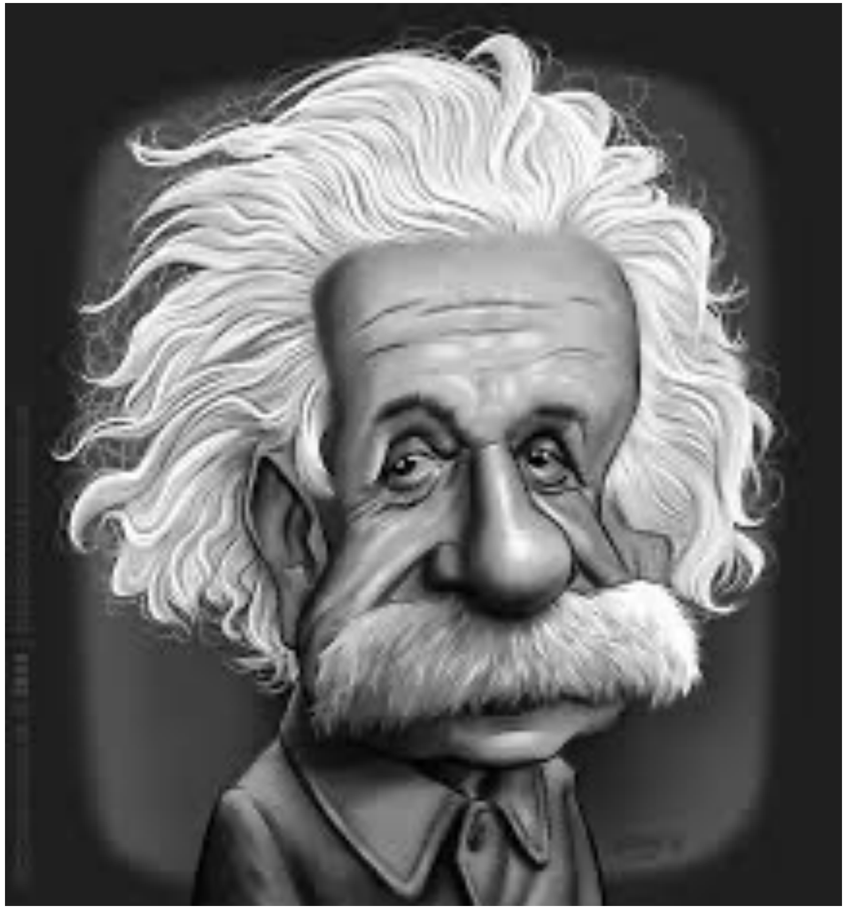
# Sistemas Dinâmicos (Socioeconômicos)

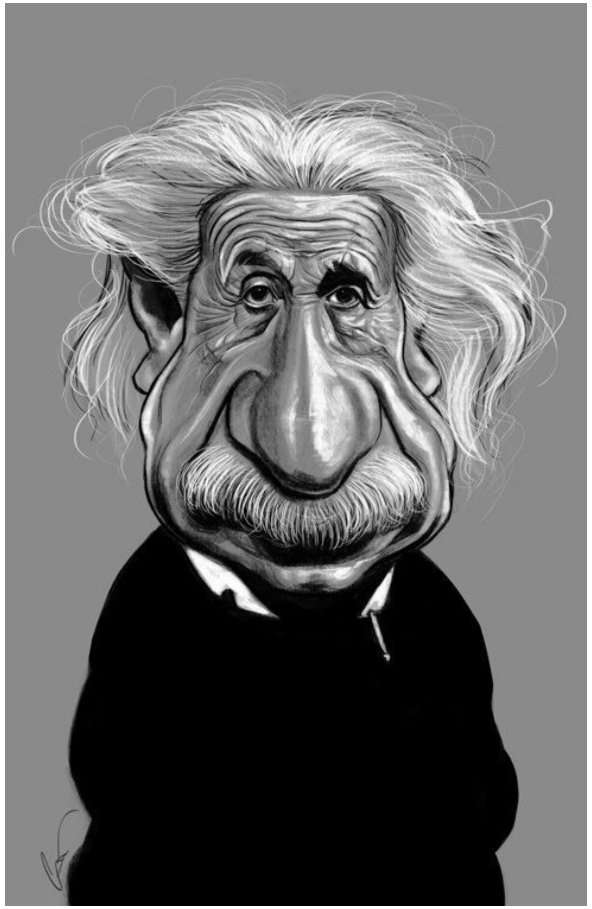




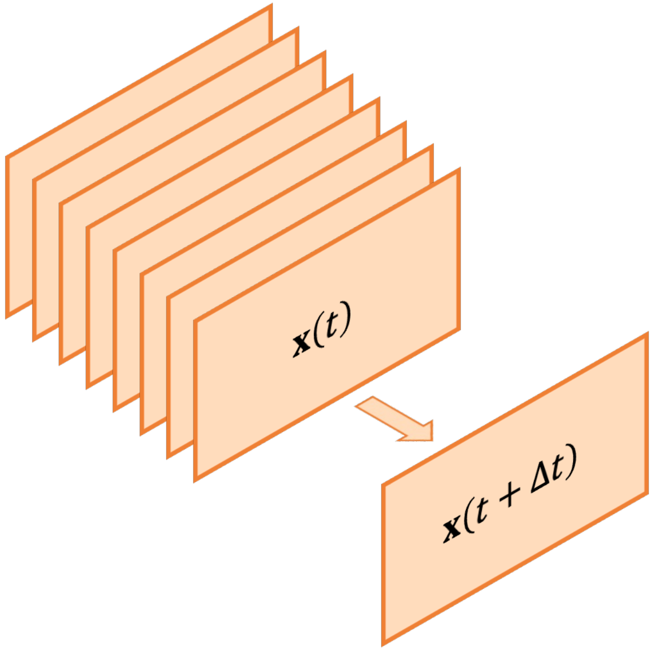
# Modelando a Realidade







# Sistema Dinâmico




→ O estado de um sistema dinâmico é definido por variáveis de estado e sua evolução no tempo é governada por uma equação de evolução.

**Tempo Discreto (Mapas)**

$$\dot{\mathbf{X}}_{n+1} = \mathbf{f}(\mathbf{X}_n); \qquad \mathbf{X} \in \mathbb{R}^{\boldsymbol{n}}$$

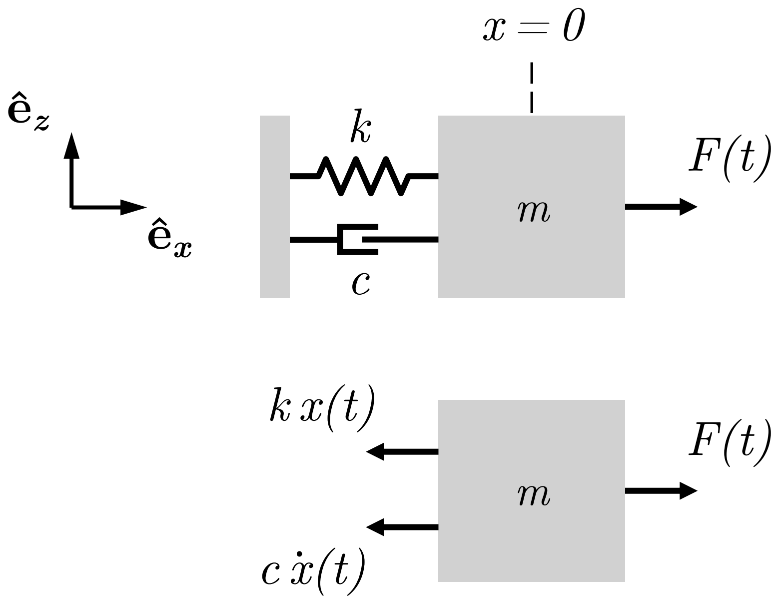
**Tempo Contínuo (EDOs)**

$$\dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t)\big); \qquad \mathbf{x}(t) \in \mathbb{R}^{\boldsymbol{n}}$$

**Tempo e Espaço Contínuos (EDPs)**

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{x}'(t), \mathbf{x}''(t), \dots); \qquad \mathbf{x}(t) \in \mathbb{R}^{\boldsymbol{n}}$$

# Sistema Dinâmico – Exemplo




$$\sum F = m\ddot{x}(t)$$

$$F(t) - kx(t) - c\dot{x}(t) = m\ddot{x}(t)$$

$$\boxed{m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)}$$

**EDO 2ª Ordem**

**Variáveis de Estado**

$$\mathbf{x}(t) = \begin{Bmatrix} x\,(t) \\ \dot{x}\,(t) \end{Bmatrix} = \begin{Bmatrix} \text{Posição} \\ \text{Velocidade} \end{Bmatrix}$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t)\big); \quad \mathbf{x}(t) \in \mathbb{R}^{\boldsymbol{n}}$$

**Sistema de EDOs de 1ª Ordem**

$$\dot{\mathbf{x}}(t) = \begin{Bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{Bmatrix} = \begin{Bmatrix} \dot{x}(t) \\ \dfrac{1}{m}\left[F(t) - kx(t) - c\dot{x}(t)\right] \end{Bmatrix}$$

# Sistema Dinâmico

## Sistema Autônomo

→ Não depende explicitamente do tempo

$$\dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t)\big); \qquad \mathbf{x}(t) \in \mathbb{R}^{\boldsymbol{n}}$$

→ Exemplo: Sistema de Lorenz

$$\dot{\mathbf{x}}(t) = \begin{Bmatrix} \sigma[y(t) - x(t)] \\ x(t)[\rho - z(t)] - y(t) \\ x(t)y(t) - \beta z(t) \end{Bmatrix}$$

## Sistema Não–Autônomo

→ Depende explicitamente do tempo

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t); \qquad \mathbf{x}(t) \in \mathbb{R}^{\boldsymbol{n}}$$

→ Exemplo: Oscilador Linear

$$\dot{\mathbf{x}}(t) = \begin{Bmatrix} \dot{x}(t) \\ \frac{1}{m}[F(t) - c\dot{x}(t) - kx(t)] \end{Bmatrix}$$

# Sistema Dinâmico Não-Autônomo em Autônomo

**Sistema Não-Autônomo**

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t); \qquad \mathbf{x}(t) \in \mathbb{R}^{\boldsymbol{n}}$$

**Sistema Autônomo**

$$\dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t)\big); \qquad \mathbf{x}(t) \in \mathbb{R}^{\boldsymbol{n+1}}$$

$$\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t), t) \\
\dot{x}_2(t) &= f_2(x_2(t), t) \\
&\vdots \\
\dot{x}_n(t) &= f_n(x_n(t), t)
\end{aligned}$$

$$\xrightarrow{x_{n+1}(t) = t}$$

$$\begin{aligned}
\dot{x}_1(t) &= f_1\big(x_1(t)\big) \\
\dot{x}_2(t) &= f_2\big(x_2(t)\big) \\
&\vdots \\
\dot{x}_n(t) &= f_n\big(x_n(t)\big) \\
\dot{x}_{n+1}(t) &= 1
\end{aligned}$$

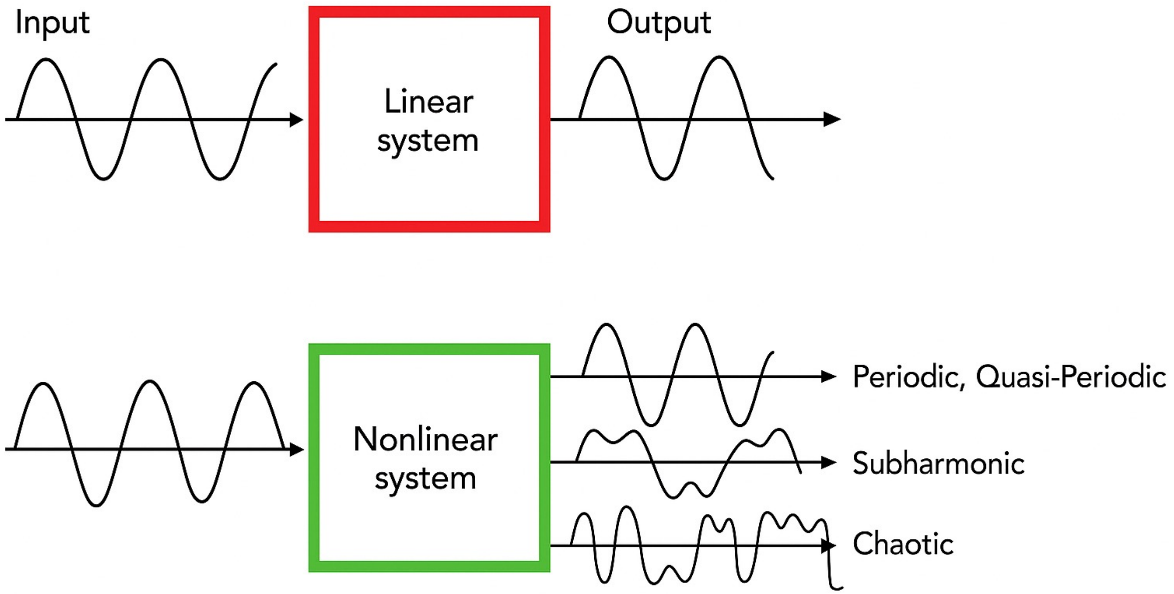
**Sistemas Equivalentes!**

# Sistemas Lineares vs Não Lineares

**Linearidade:**

$$f(x+y) = f(x) + f(y) \rightarrow \text{Aditividade}$$

$$f(\alpha x) = \alpha f(x) \rightarrow \text{Homogeneidade}$$




→ **Pequenas causas estão relacionadas a pequenos efeitos (Princípio da Superposição)**

**Não-Linearidade:**

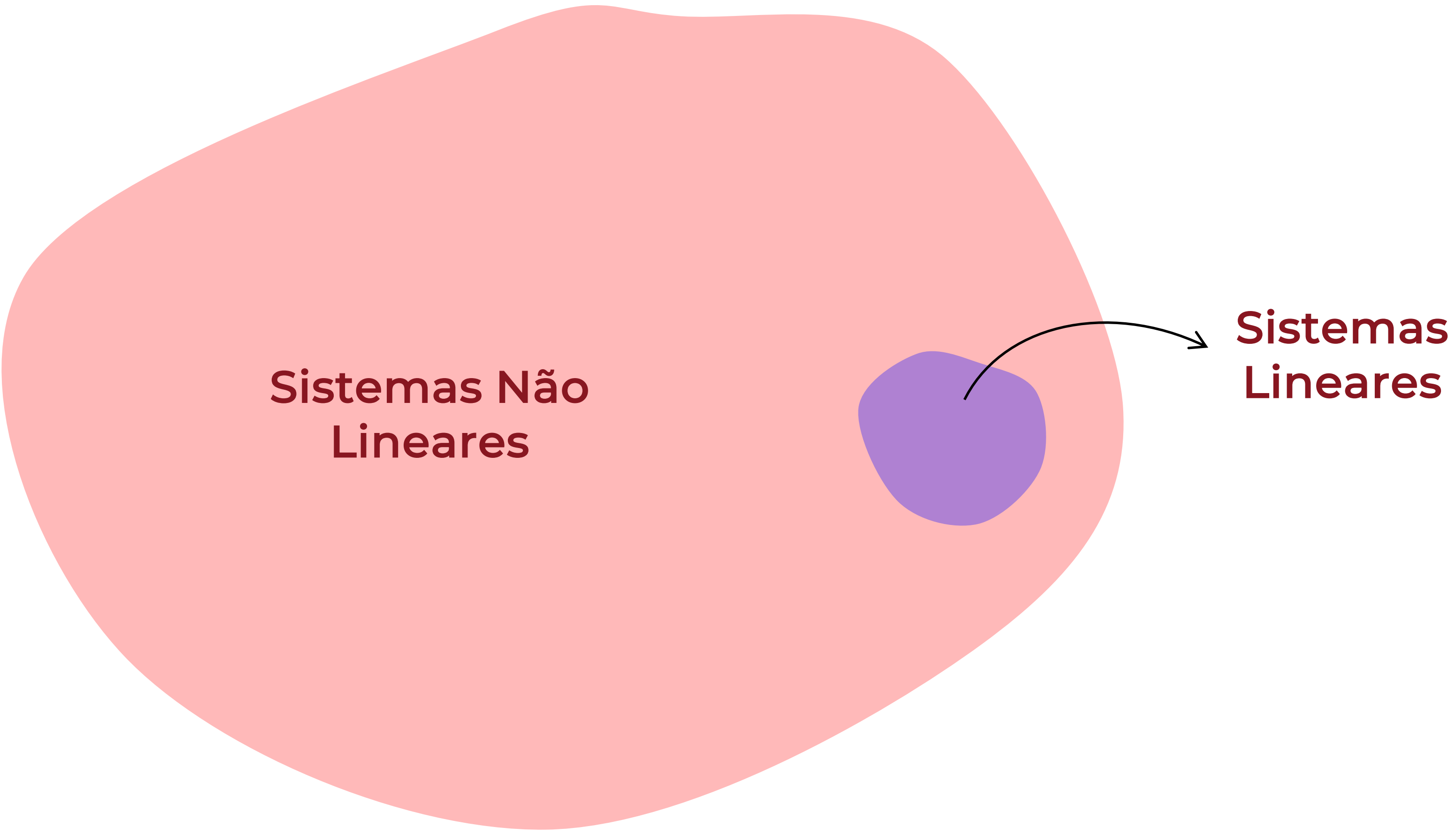
$$x^n \mid \sin(x) \mid x(t)y(t) \mid \ldots$$

→ **Em geral, sistemas dinâmicos não lineares não possuem solução analítica.**

→ **Apresentam soluções estáveis e instáveis**

→ **Cada solução é única!**

# Sistemas Dinâmicos Lineares e Não Lineares

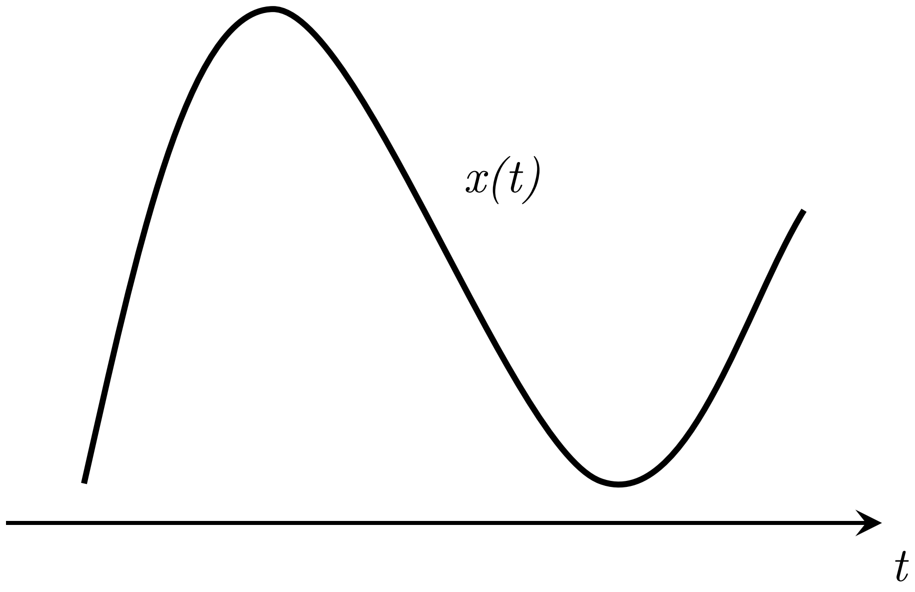


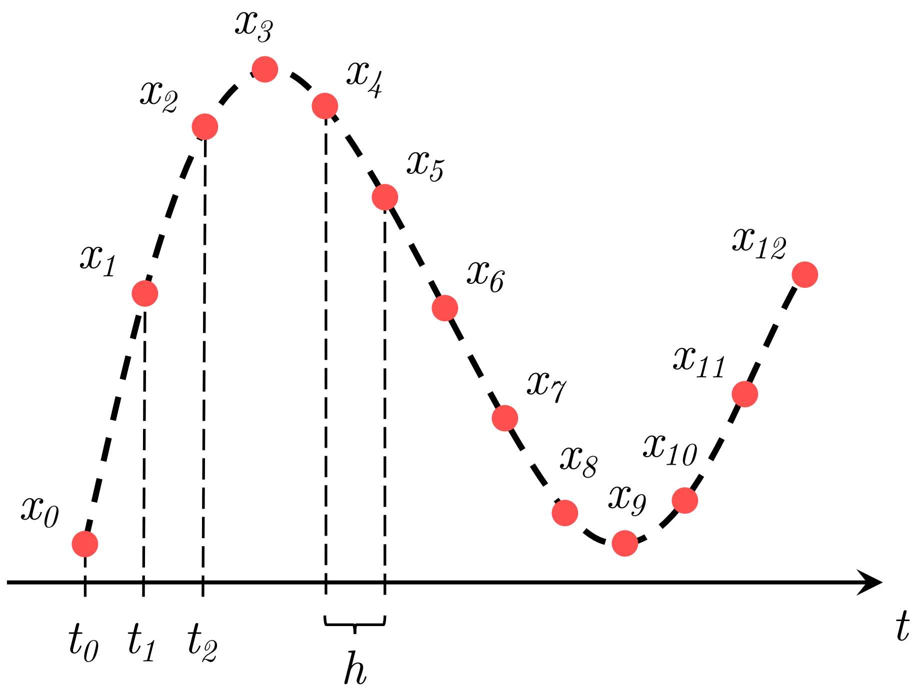
# Métodos de Solução Numérica

→ **Descrevem o problema real, contínuo no tempo, através de uma discretização.**

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t); \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

$$t_{i+1} = t_i + h$$



# Métodos de Solução Numérica

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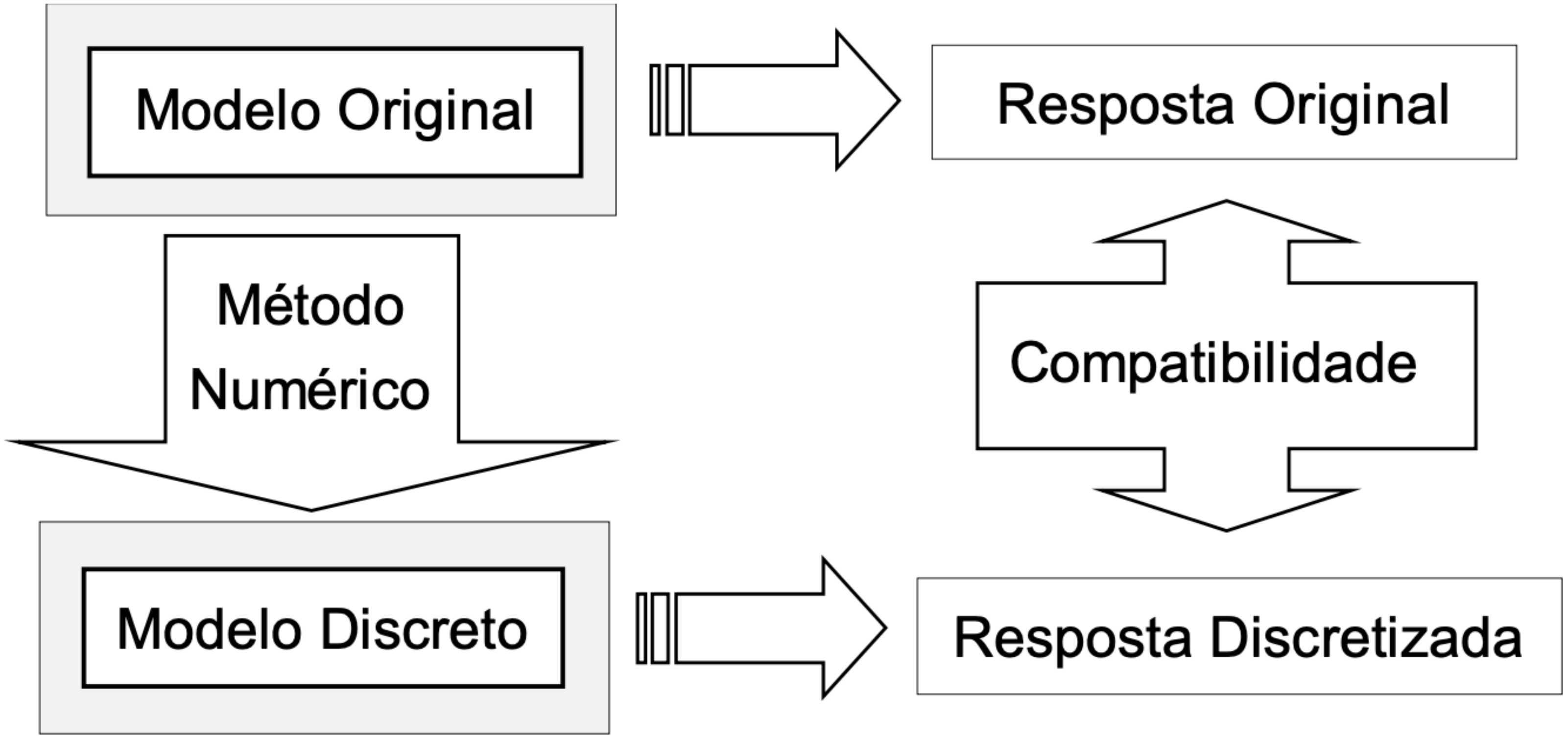


→ **Força Bruta (Apenas soluções estáveis)**

# Ideia Geral

→ **Problema de valor inicial:**

$$\dot{x} = f(x,t); \qquad x(t_0) = x_0$$

→ **Discretização:**

$$t_{i+1} = t_i + h$$

→ **O valor $x(t_{i+1})$ pode ser obtido por expansão em série de Taylor:**

$$x(t_{i+1}) = x(t_i) + \dot{x}(t_i)h + \ddot{x}(t_i)\frac{h^2}{2!} + \dddot{x}(t_i)\frac{h^3}{3!} + \ddddot{x}(t_i)\frac{h^4}{4!} + \cdots$$

→ **É possível escrever as derivadas temporais de $x$ em função de $f(x,t)$:**

$$\dot{x} = f(x,t)$$

$$\ddot{x} = \frac{d}{dt}[f(x,t)] = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial t} = f'f + \dot{f}$$

# Ideia Geral

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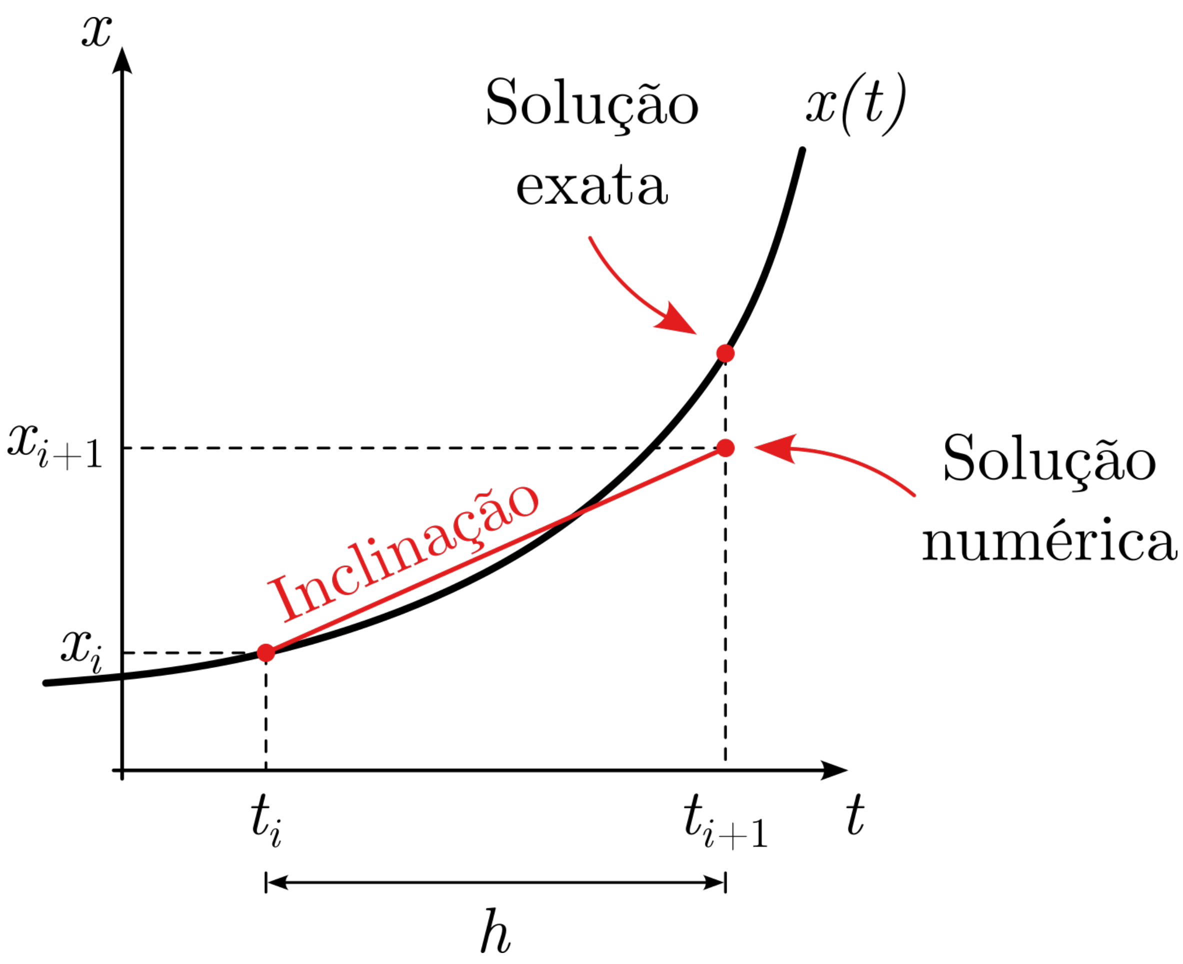
$$\dddot{x} = \frac{d}{dt}[f'f + \dot{f}] = \frac{\partial}{\partial x}[f'f + \dot{f}]\frac{\partial x}{\partial t} + \frac{\partial}{\partial t}[f'f + \dot{f}]\frac{\partial t}{\partial t} = \ddot{f} + f'\dot{f} + 2\dot{f}'f + (f')^2 f + f''(f)^2$$

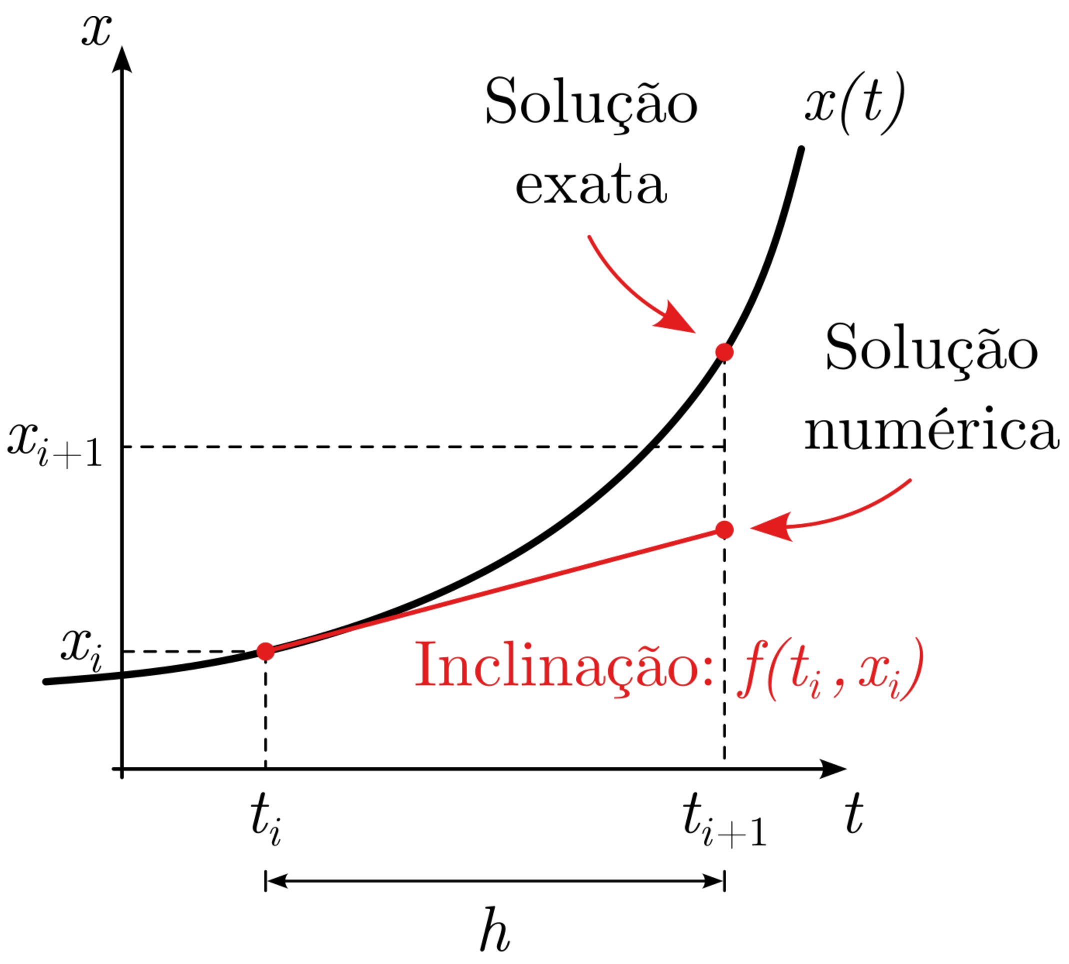
$$\ddddot{x} = \dddot{f} + 3\ddot{f}'f + f'''(f)^3 + \ddot{f}f' + 3f''\dot{f}f + 3\dot{f}''(f)^2 + 5\dot{f}'f'f + 4f''f'(f)^2 + 3\dot{f}'\dot{f} + (f')^2\dot{f} + (f')^3 f$$

# Métodos Explícitos




$$t_{i+1} = t_i + h$$
$$x_{i+1} = x_i + \text{Inclinação} \cdot h$$

# Métodos Explícito de Euler

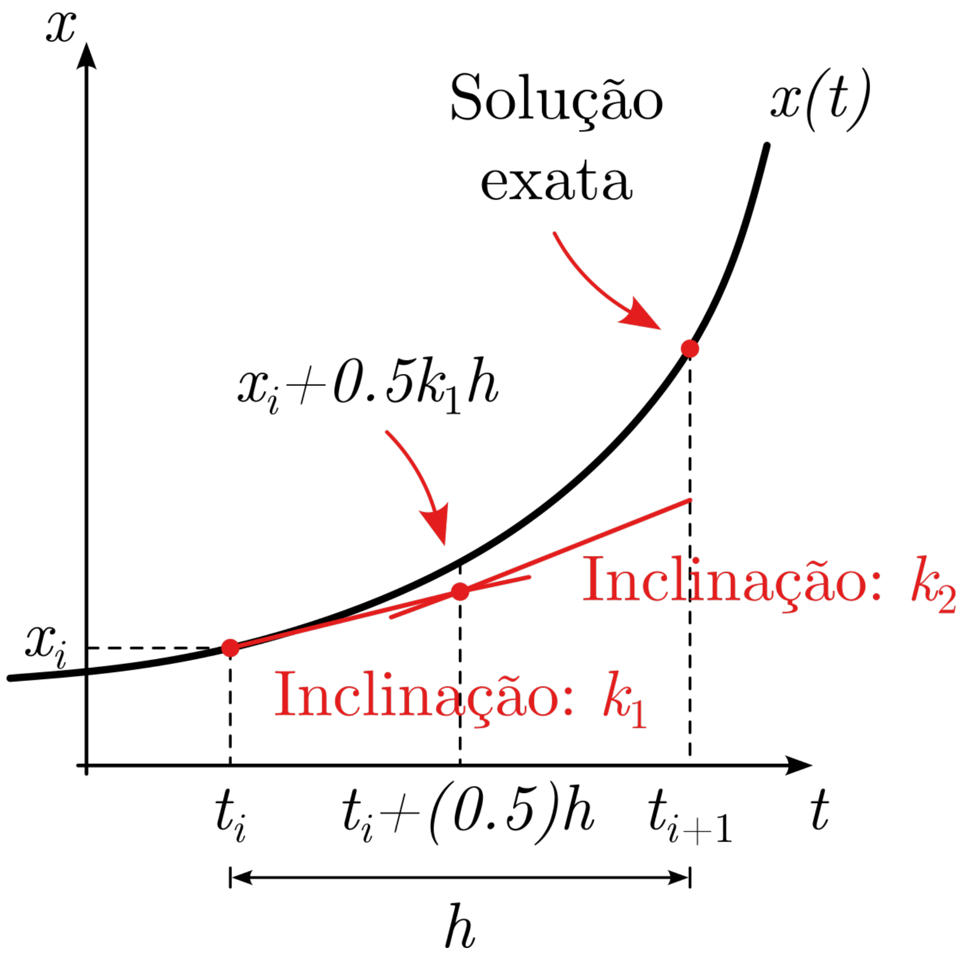



$$\text{Inclinação} = \left.\frac{dx}{dt}\right|_{t=t_i} = f(t_i, x_i)$$

$$t_{i+1} = t_i + h$$

$$x_{i+1} = x_i + f(t_i, x_i) \cdot h$$

# Runge-Kutta de 2ª Ordem



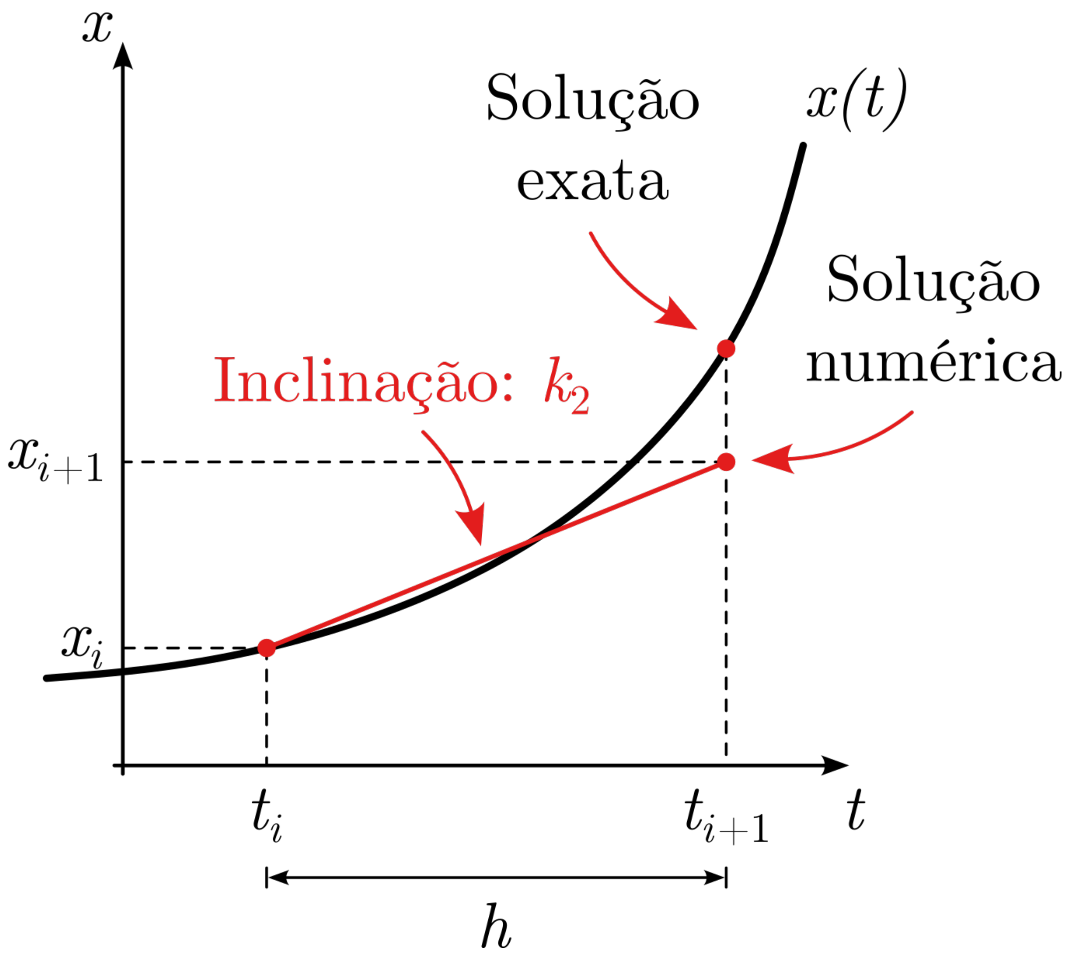

$$t_{i+1} = t_i + h$$

$$k_1 = f(t_i, x_i)$$

$$k_2 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_1h\right)$$

$$x_{i+1} = x_i + k_2h$$

# Runge-Kutta de 2ª Ordem



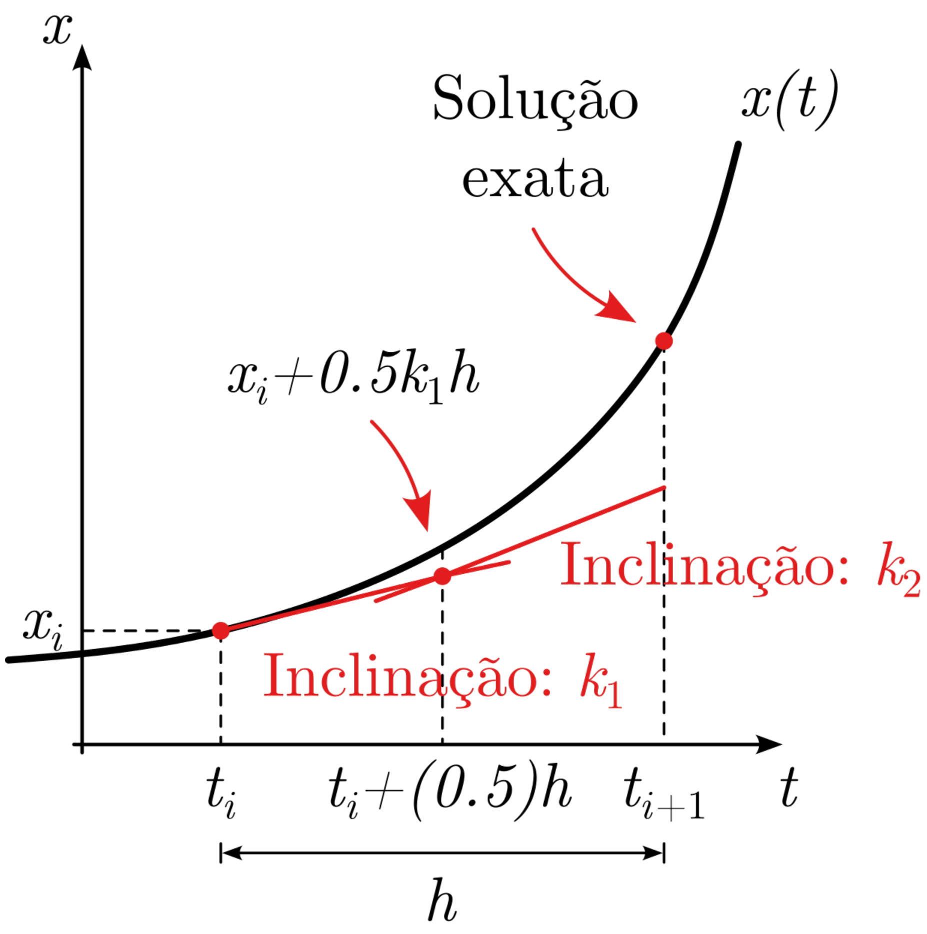

$$t_{i+1} = t_i + h$$

$$k_1 = f(t_i, x_i)$$

$$k_2 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_1 h\right)$$

$$x_{i+1} = x_i + k_2 h$$

# Runge-Kutta de 4ª Ordem




$$t_{i+1} = t_i + h$$

$$k_1 = f(t_i, x_i)$$

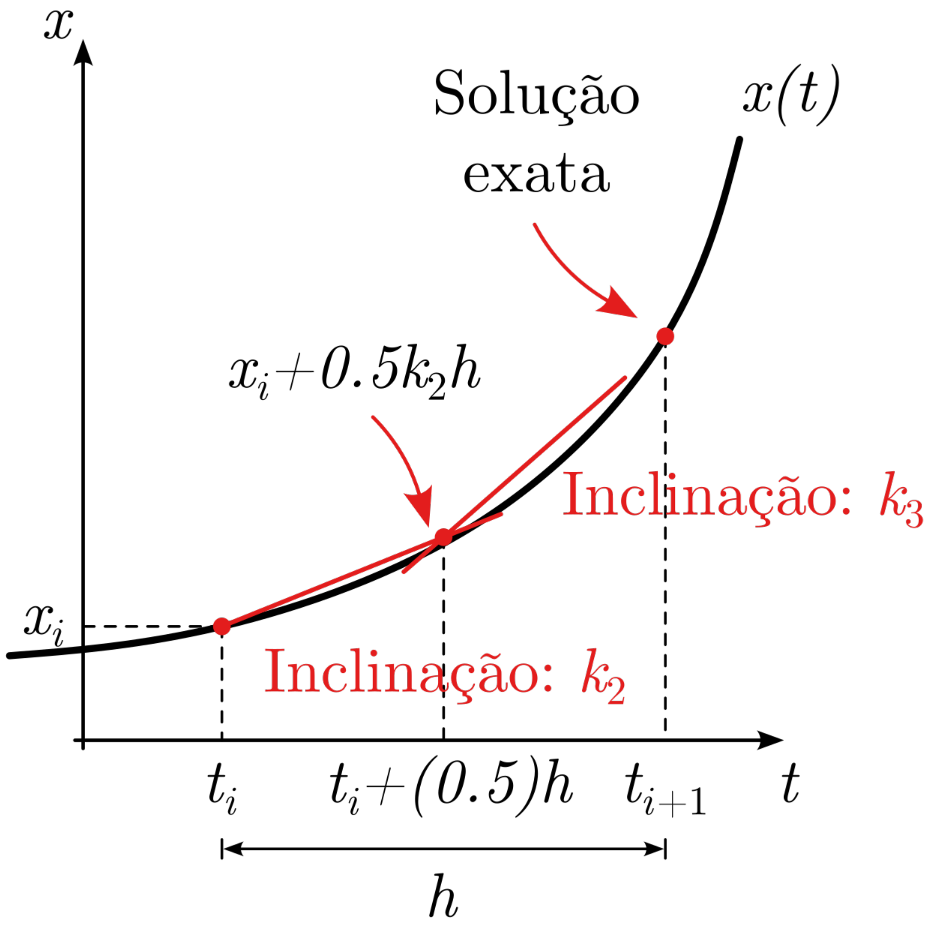
$$k_2 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_1h\right)$$

$$k_3 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_2h\right)$$

$$k_4 = f(t_i + h, x_i + k_3h)$$

$$x_{i+1} = x_i + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$

# Runge-Kutta de 4ª Ordem




$$t_{i+1} = t_i + h$$

$$k_1 = f(t_i, x_i)$$

$$k_2 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_1h\right)$$

$$k_3 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_2h\right)$$

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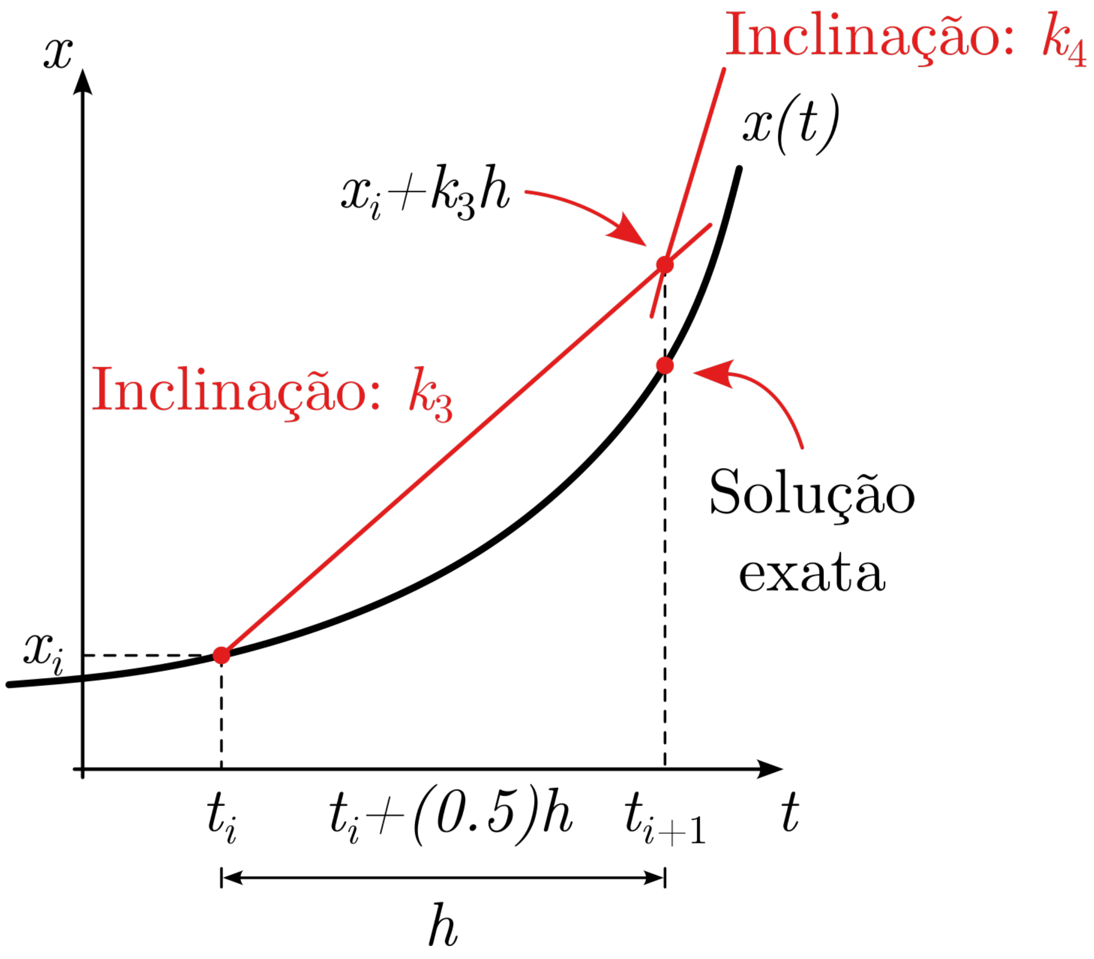
# Runge-Kutta de 4ª Ordem




$$t_{i+1} = t_i + h$$

$$k_1 = f(t_i, x_i)$$

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$$k_3 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_2 h\right)$$

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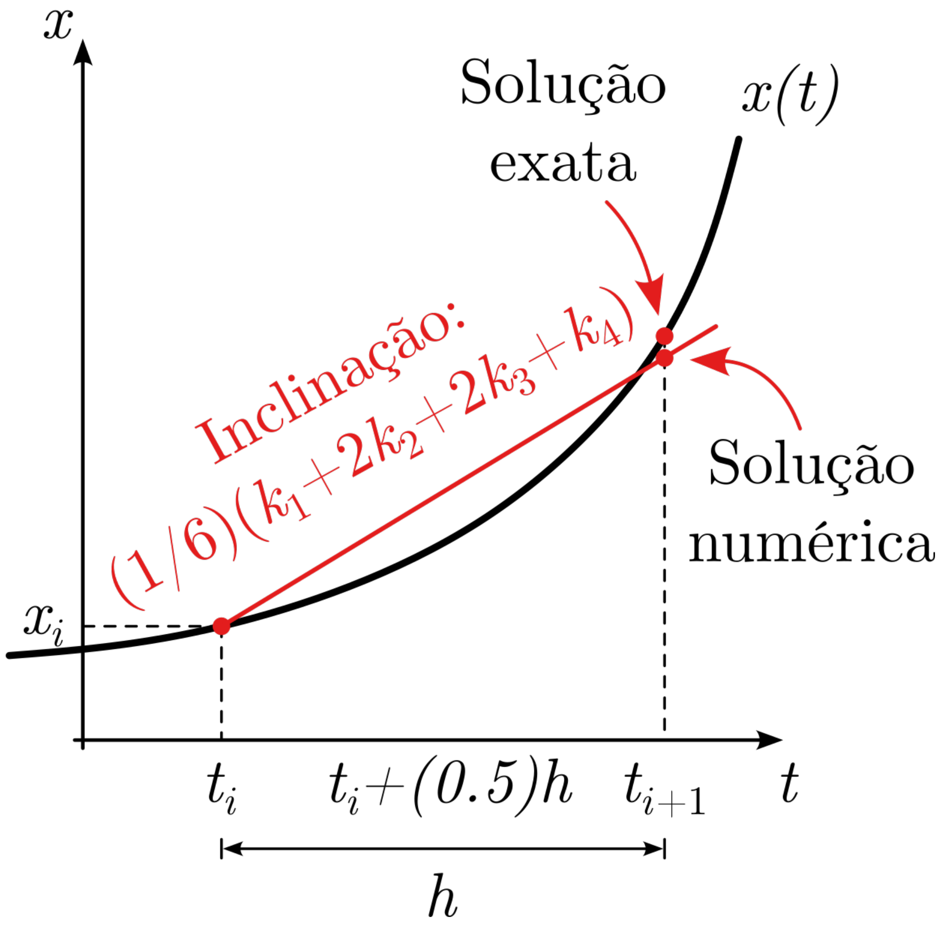
## Runge-Kutta de 4ª Ordem




$$t_{i+1} = t_i + h$$

$$k_1 = f(t_i, x_i)$$

$$k_2 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f(t_i + h, x_i + k_3 h)$$

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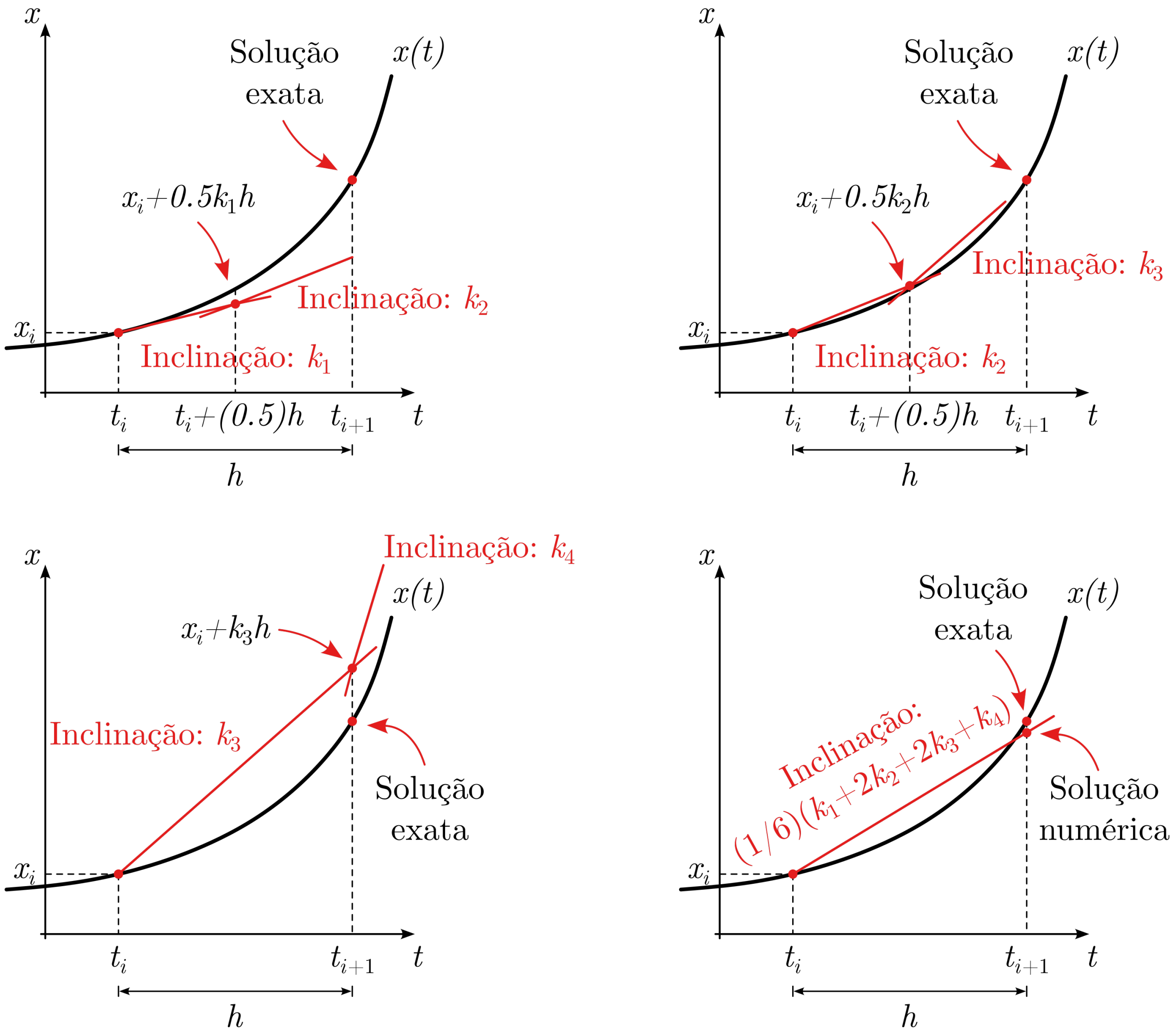
# Runge-Kutta de 4ª Ordem




$$t_{i+1} = t_i + h$$

$$k_1 = f(t_i, x_i)$$
$$k_2 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_1h\right)$$
$$k_3 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_2h\right)$$
$$k_4 = f(t_i + h, x_i + k_3h)$$

$$x_{i+1} = x_i + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$

# Generalização dos Métodos Runge-Kutta Explícitos

$$t_{i+1} = t_i + h$$

$$x_{i+1} = x_i + h \sum_{j=1}^{s} b_j k_j$$

$$\begin{aligned}
k_1 &= f(t_i, x_i), \\
k_2 &= f\big(t_i + c_2 h, x_i + h(a_{21} k_1)\big), \\
k_3 &= f\big(t_i + c_3 h, x_i + h(a_{31} k_1 + a_{32} k_2)\big), \\
&\vdots \\
k_s &= f\Big(t_i + c_s h, x_i + h\big(a_{s1} k_1 + a_{s2} k_2 + \cdots + a_{s,s-1} k_{s-1}\big)\Big)
\end{aligned}$$

**Tabela de Butcher**

$$\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & & \ddots & & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & \\
\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}$$

# Generalização dos Métodos Runge-Kutta Explícitos

## Método Explícito de Euler

$$x_{i+1} = x_i + f(t_i, x_i) \cdot h$$

$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array}$$

## Runge-Kutta de 2ª Ordem

$$k_1 = f(t_i, x_i)$$
$$k_2 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_1 h\right)$$
$$x_{i+1} = x_i + k_2 h$$

$$\begin{array}{c|cc} 0 & & \\ 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

## Runge-Kutta de 4ª Ordem

$$k_1 = f(t_i, x_i)$$
$$k_2 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_1 h\right)$$
$$k_3 = f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}k_2 h\right)$$
$$k_4 = f(t_i + h, x_i + k_3 h)$$
$$x_{i+1} = x_i + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

# Métodos Runge-Kutta Explícitos Adaptativos

→ **Métodos que estimam o erro de truncamento durante a execução, e adaptam (diminuem ou aumentam) o passo de acordo com o erro.**

$$t_{i+1} = t_i + h$$

$$x_{i+1} = x_i + h \sum_{j=1}^{s} b_j k_j \quad \text{(Passo de Ordem } p\text{)}$$

$$x_{i+1}^* = x_i + h \sum_{j=1}^{s} b_j^* k_j \quad \text{(Passo de Ordem } p-1\text{)}$$

$$e_{i+1} = x_{i+1} - x_{i+1}^* = h \sum_{j=1}^{s} (b_j - b_j^*) k_j$$

**Tabela de Butcher**

$$\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & & \ddots & & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & \\
\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s \\
 & b_1^* & b_2^* & \cdots & b_{s-1}^* & b_s^*
\end{array}$$

# Método Runge-Kutta-Fehlberg (Ordem 4 e 5)

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | | | | | | |
| 1/4 | 1/4 | | | | | |
| 3/8 | 3/32 | 9/32 | | | | |
| 12/13 | 1932/2197 | –7200/2197 | 7296/2197 | | | |
| 1 | 439/216 | –8 | 3680/513 | -845/4104 | | |
| 1/2 | –8/27 | 2 | –3544/2565 | 1859/4104 | –11/40 | |
| | 16/135 | 0 | 6656/12825 | 28561/56430 | –9/50 | 2/55 |
| | 25/216 | 0 | 1408/2565 | 2197/4104 | –1/5 | 0 |

# Método Runge-Kutta-Dormand-Prince (Ordem 4 e 5)

Matlab
ODE45

Octave
ODE45

Scipy
SOLVEIVP

Julia
DIfferentialEquations.jl

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | | | | | | | |
| 1/5 | 1/5 | | | | | | |
| 3/10 | 3/40 | 9/40 | | | | | |
| 4/5 | 44/45 | −56/15 | 32/9 | | | | |
| 8/9 | 19372/6561 | −25360/2187 | 64448/6561 | −212/729 | | | |
| 1 | 9017/3168 | −355/33 | 46732/5247 | 49/176 | −5103/18656 | | |
| 1 | 35/384 | 0 | 500/1113 | 125/192 | −2187/6784 | 11/84 | |
| | 35/384 | 0 | 500/1113 | 125/192 | −2187/6784 | 11/84 | 0 |
| | 5179/57600 | 0 | 7571/16695 | 393/640 | −92097/339200 | 187/2100 | 1/40 |

# Outros Métodos Relevantes

**Métodos de Ordem Superior (Maior Precisão)**

→ Runge-Kutta-Felhberg 7-8 (RKF78)

→ Runge-Kutta-Dormand-Prince (RKDP78)

**Métodos Implícitos (Para Sistemas Rígidos)**

→ Runge-Kutta Implícitos

→ Runge-Kutta Diagonalmente Implícitos (DIRK)

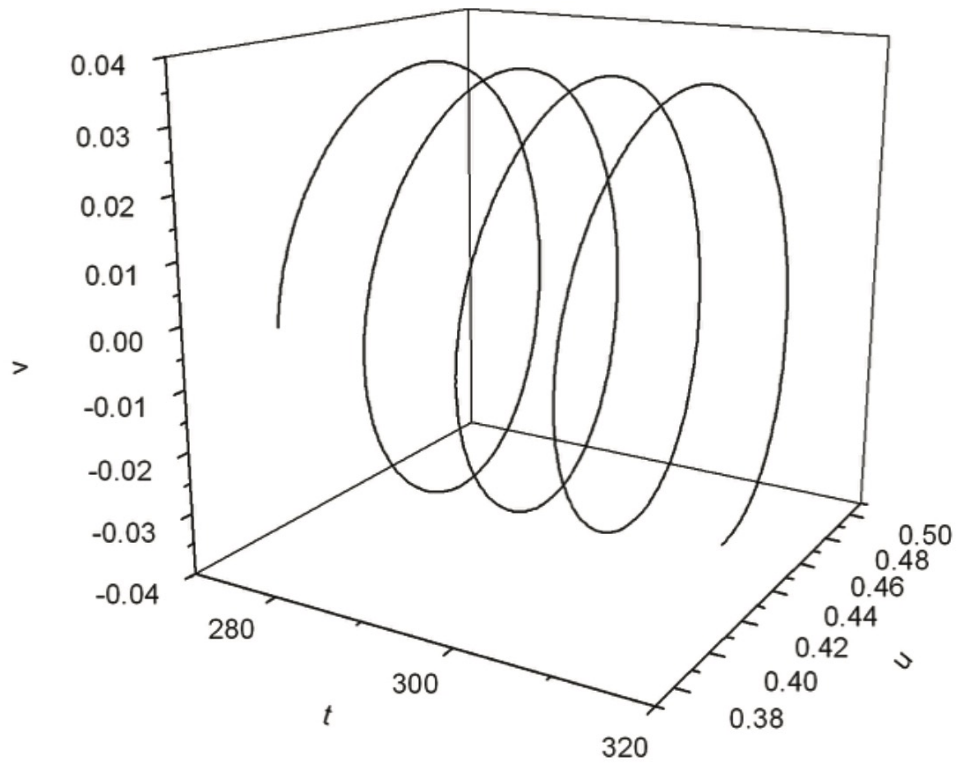
**Métodos Preservam a Estrutura do Sistema (Mantém Energia e Estrutura Geométrica)**

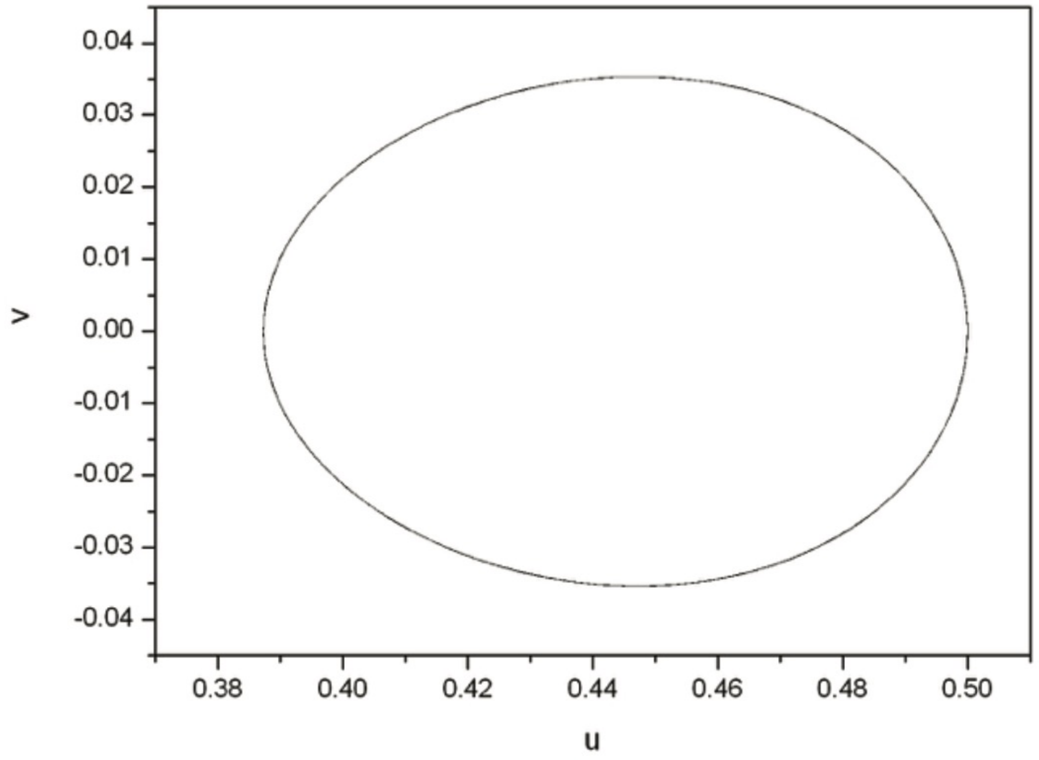
→ Integratores Simpléticos (Sistemas Hamiltonianos)

→ Integradores Variacionais (Equações de Euler-Lagrange)

# Espaço de Fase

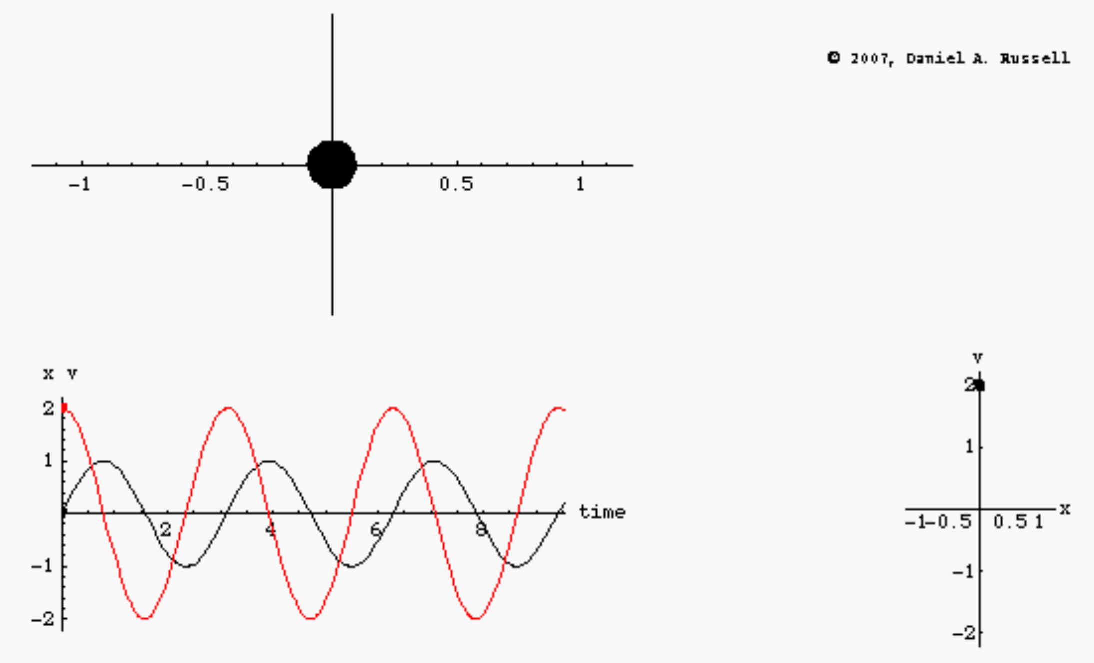
→ Espaço formado pelas variáveis dependentes do sistema (coordenadas generalizadas / graus de liberdade)

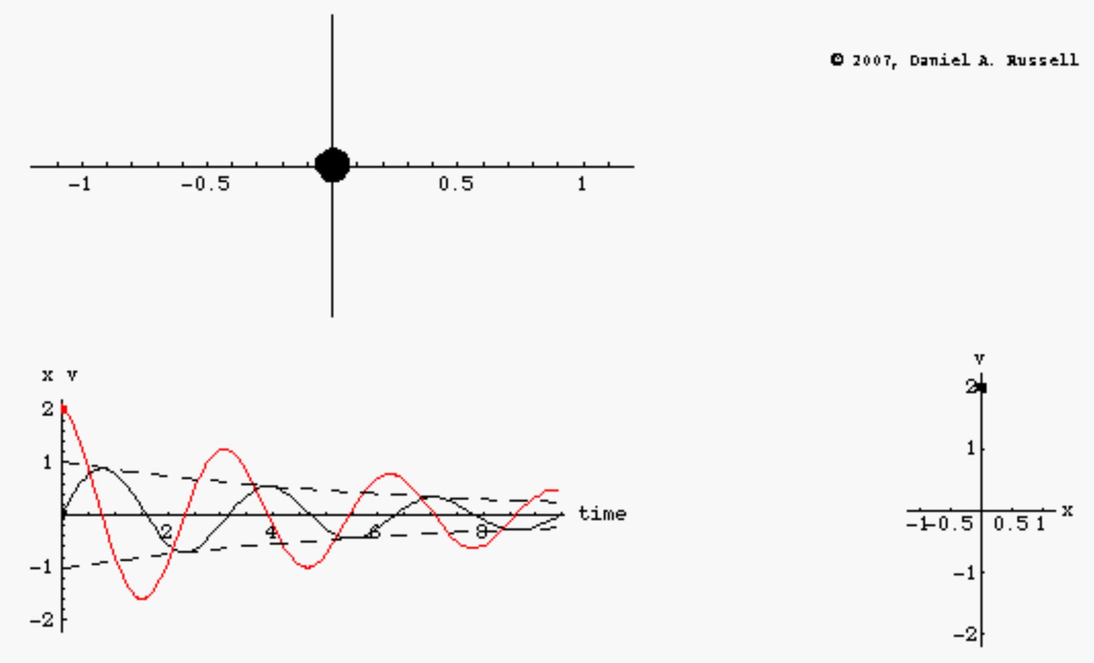


# Espaço de Fase

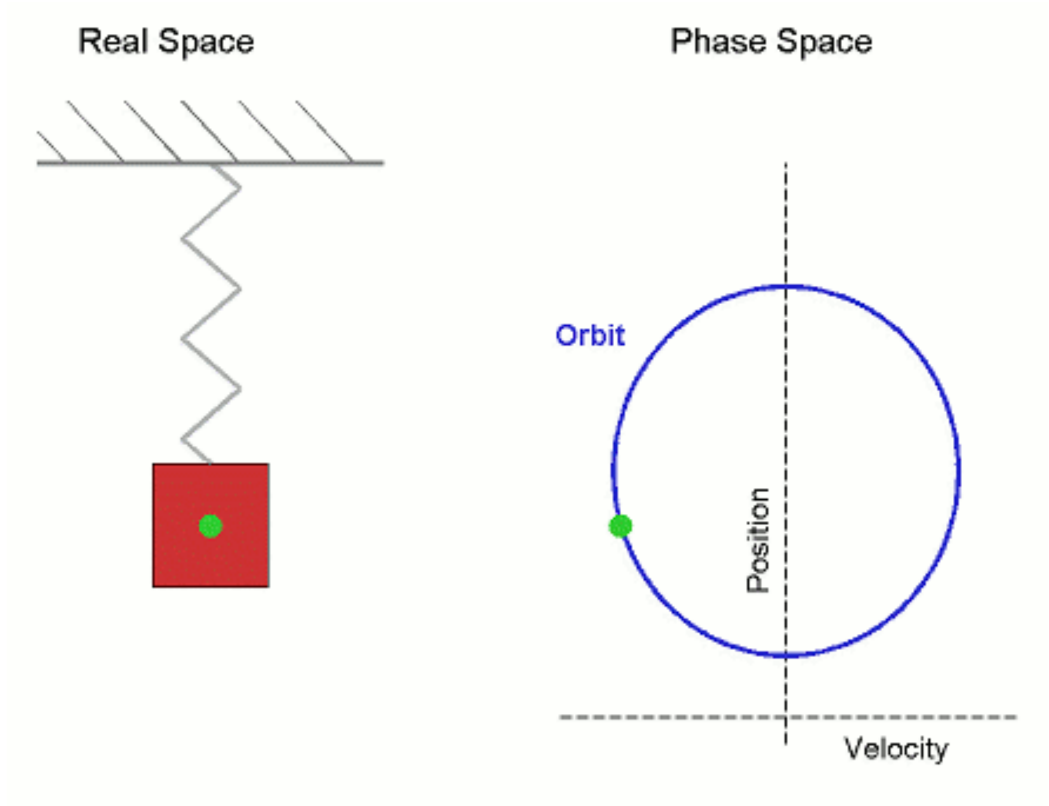
- Espaço formado pelas variáveis dependentes do sistema (coordenadas generalizadas / graus de liberdade)

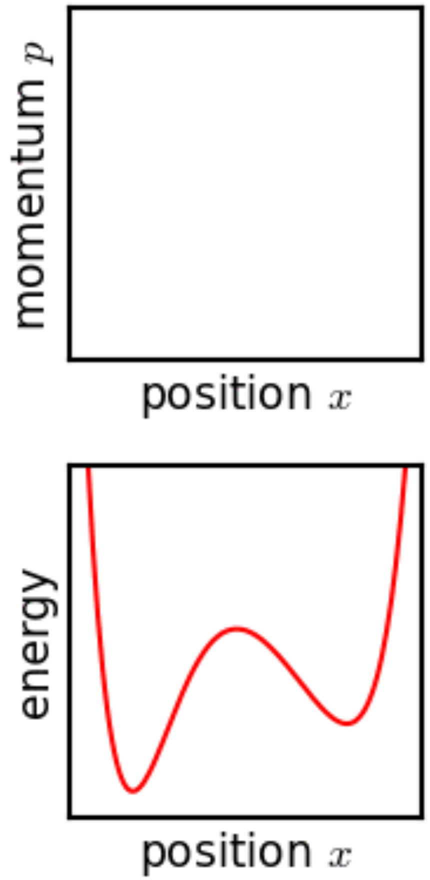


# Espaço de Fase

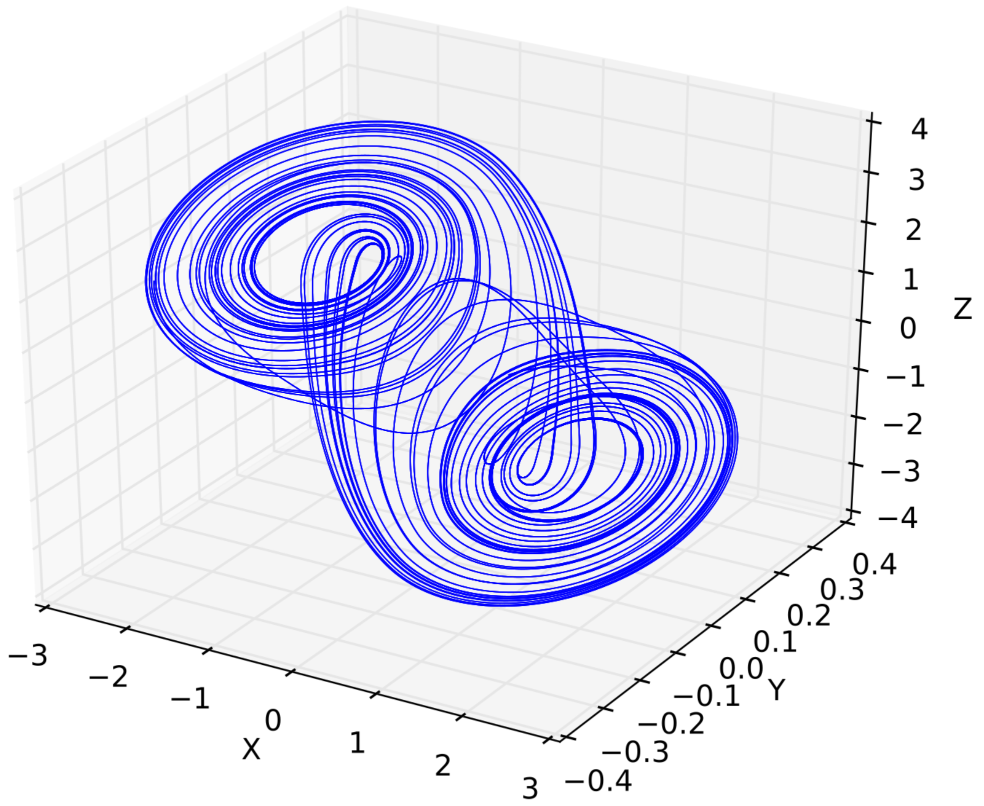
→ Espaço formado pelas variáveis dependentes do sistema (coordenadas generalizadas / graus de liberdade)

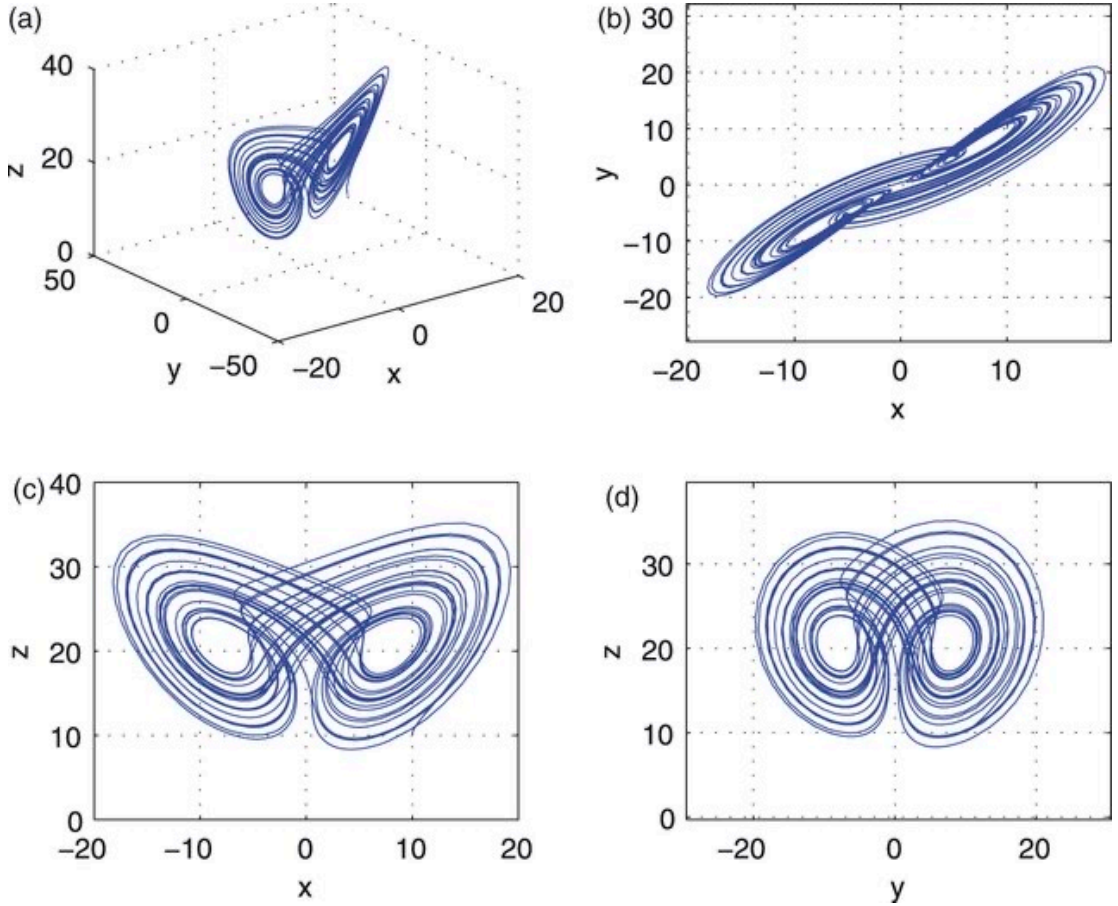


# Espaço de Fase (Espaço vs Subespaço)

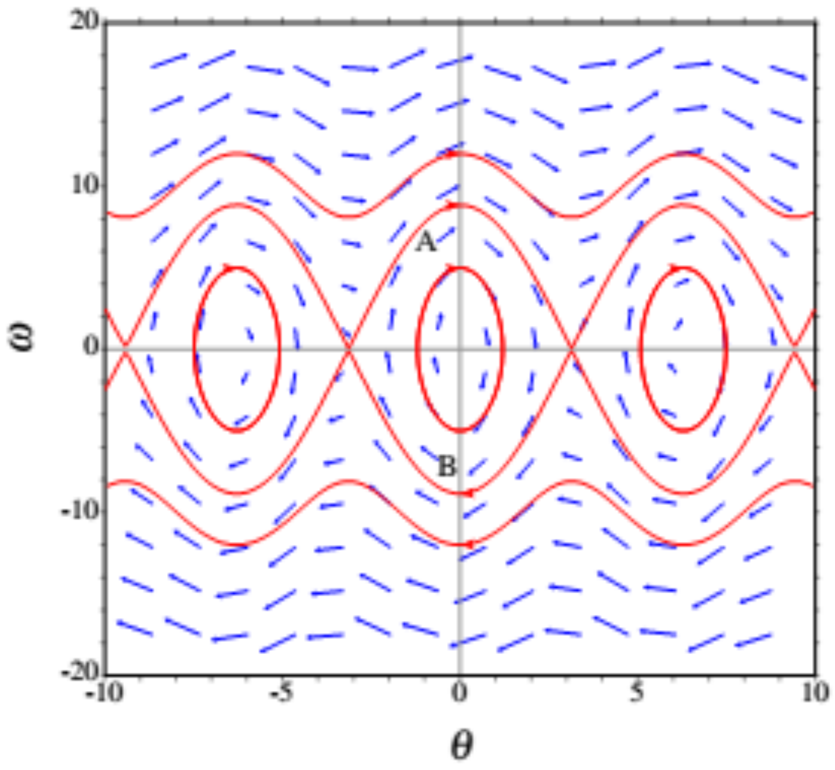
→ Espaço formado pelas variáveis dependentes do sistema (coordenadas generalizadas / graus de liberdade)



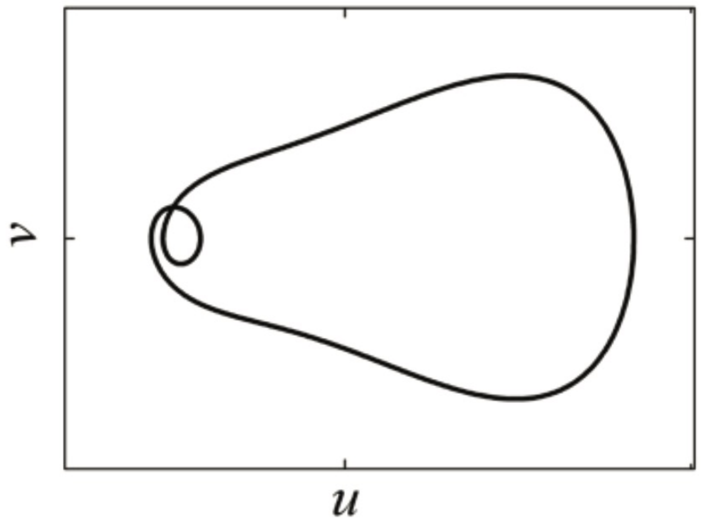
# Espaço de Fase (Campo Vetorial)

→ Espaço formado pelas variáveis dependentes do sistema (coordenadas generalizadas / graus de liberdade)

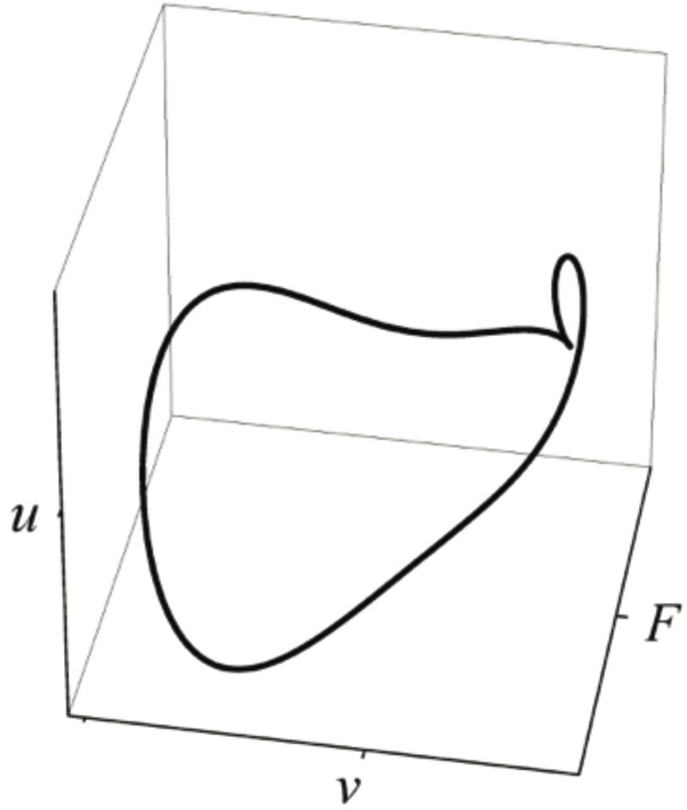


# Tipos de Visualização Podem Limitar a Compreensão do Sistema

## Sistema Não–Autônomo

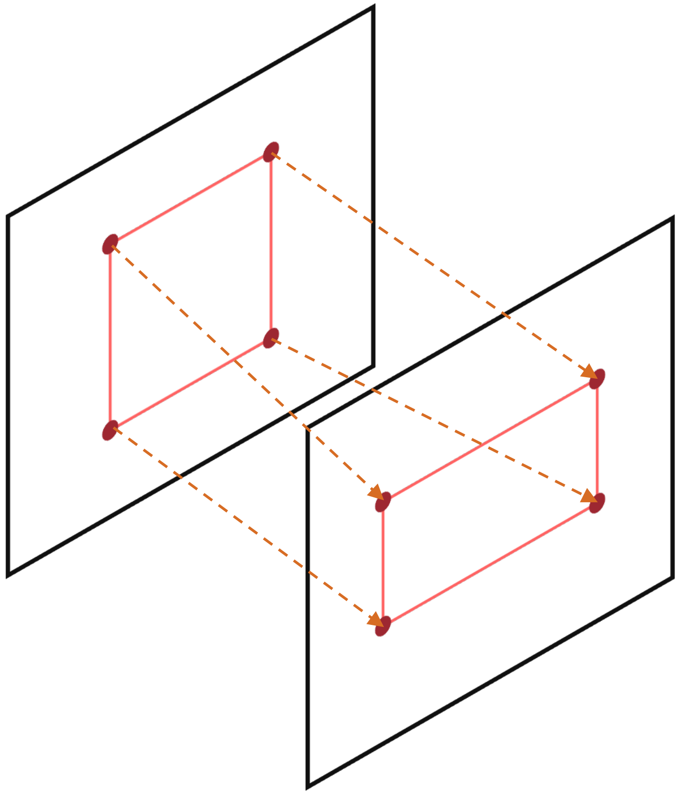


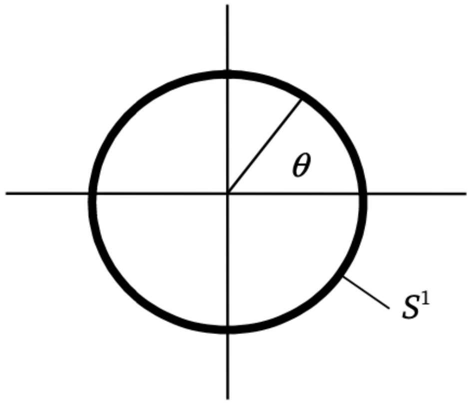

## Sistema Autônomo Equivalente

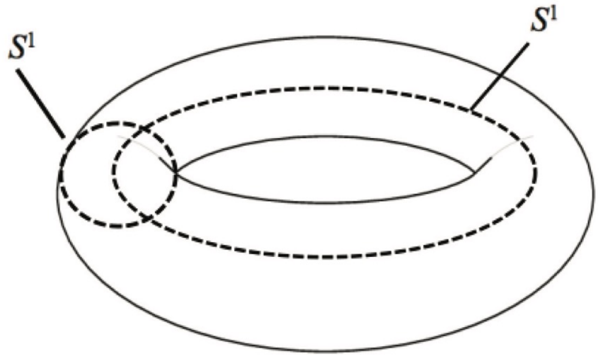


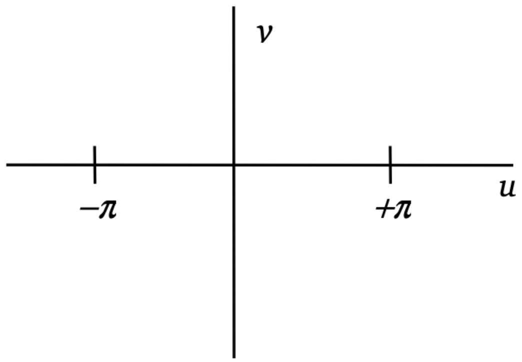
# Topologia

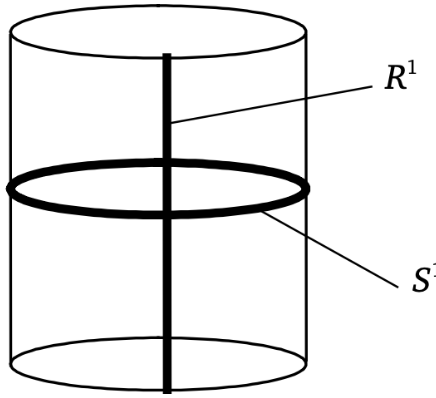
$$\dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t)\big); \qquad \mathbf{x}(t) \in \mathbb{R}^{n}$$



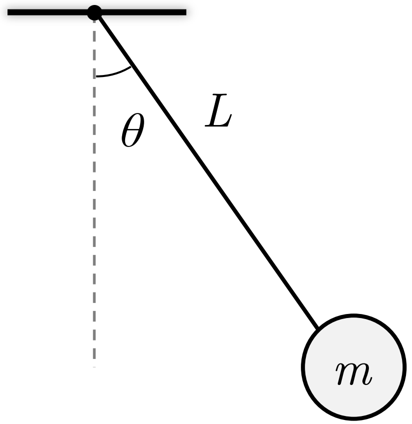


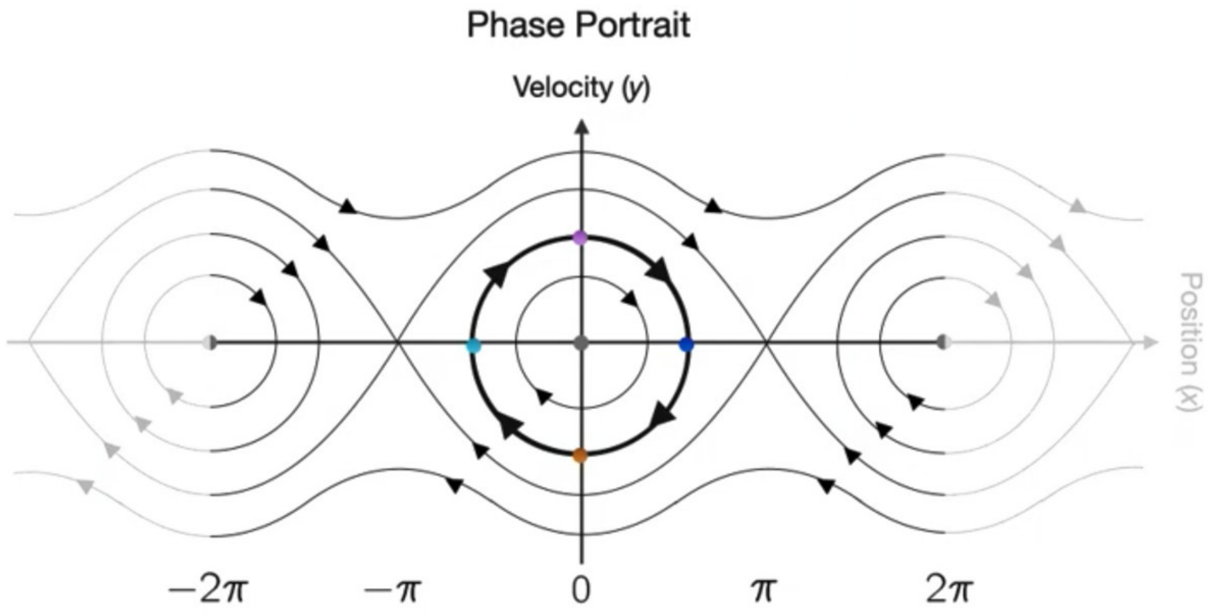



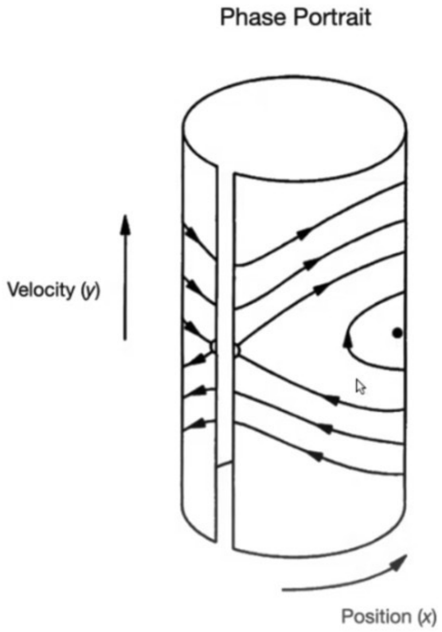
# Topologia



$\theta$
$L$
$m$

Phase Portrait
Velocity (y)
Position (x)
$-2\pi$
$-\pi$
0
$\pi$
$2\pi$

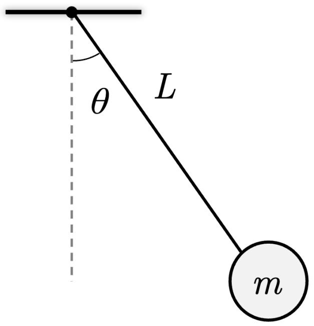
Phase Portrait
Velocity (y)
Position (x)

# Sistemas Lineares vs Não Lineares (Exemplo)

## VIBRAÇÕES LIVRES SEM AMORTECIMENTO

Ângulo Inicial: $\theta_0 = -10\,°$

Velocidade Inicial: $\dot{\theta}_0 = 0\,°/s$

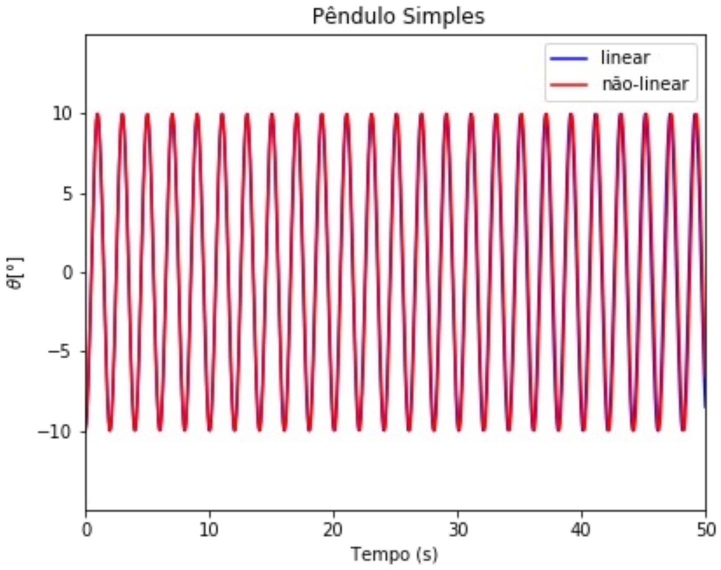


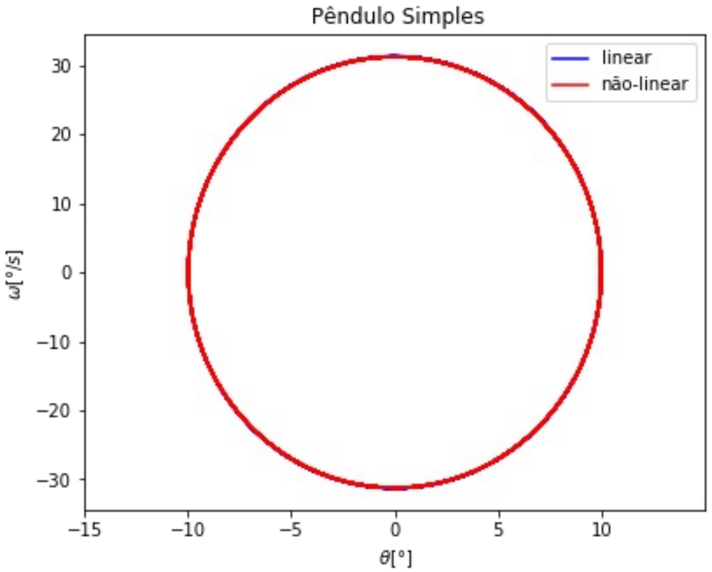

### Não Linear

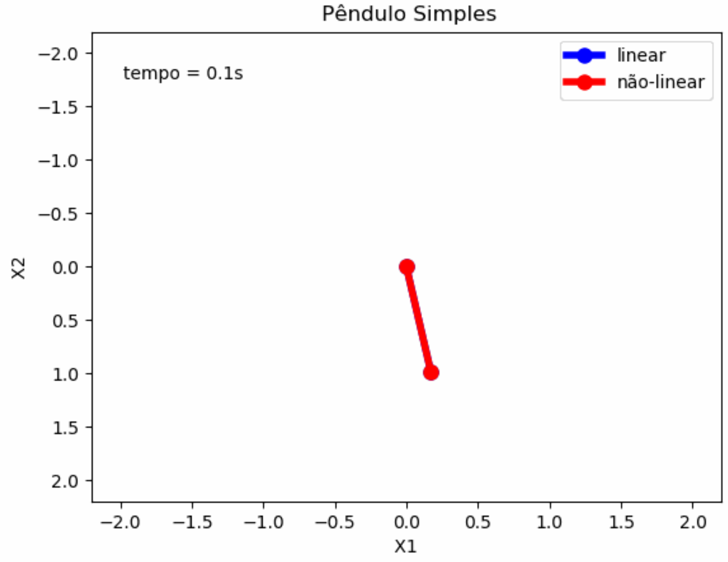
$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2 \sin(\theta) = f(t)}$$

### Linear

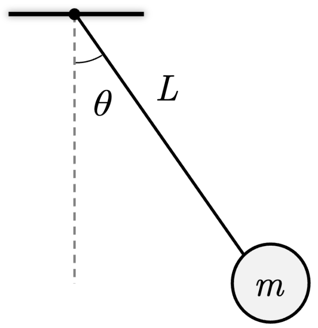
$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2 \theta = f(t)}$$



# Sistemas Lineares vs Não Lineares (Exemplo)




**Não Linear**

$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2 \sin(\theta) = f(t)}$$

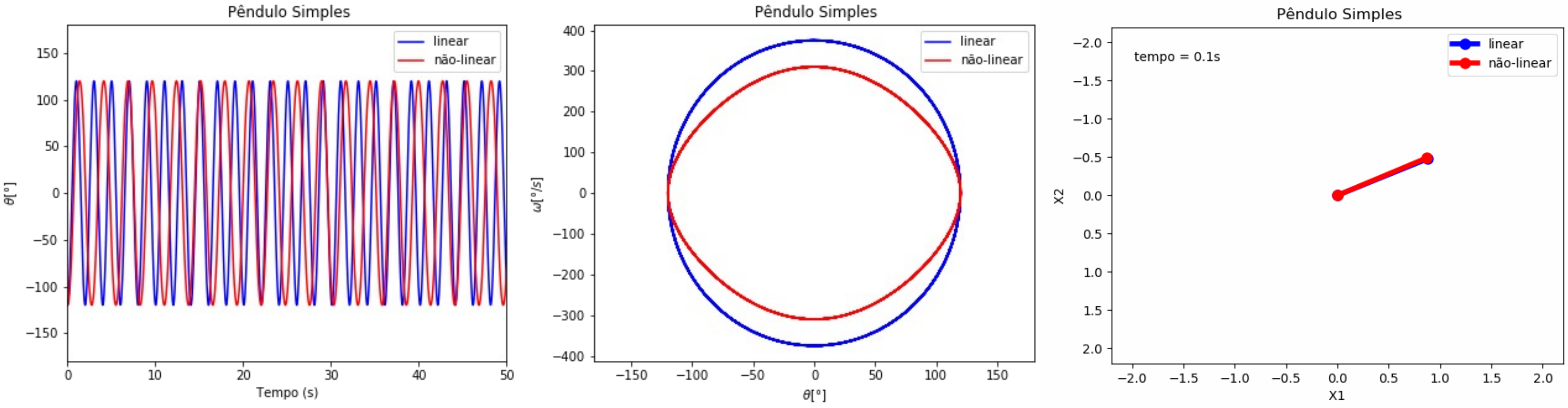
**Linear**

$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2 \theta = f(t)}$$

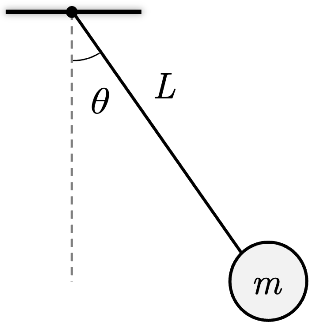
## VIBRAÇÕES LIVRES SEM AMORTECIMENTO

Ângulo Inicial: $\theta_0 = -120\,°$

Velocidade Inicial: $\dot{\theta}_0 = 0\,°/s$



# Sistemas Lineares vs Não Lineares (Exemplo)




## Não Linear

$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2 \sin(\theta) = f(t)}$$

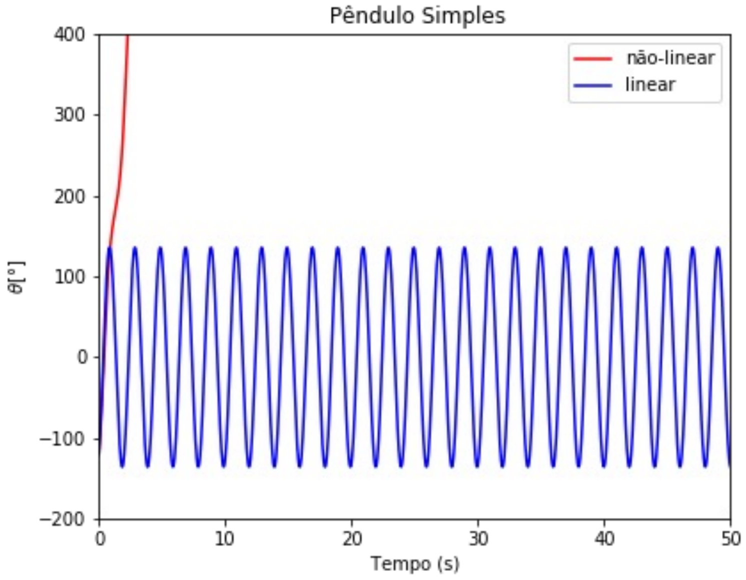
## Linear

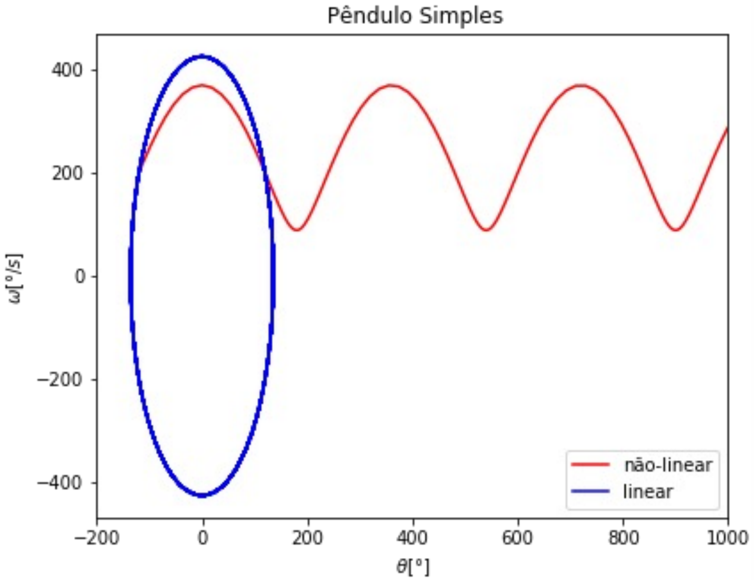
$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = f(t)}$$

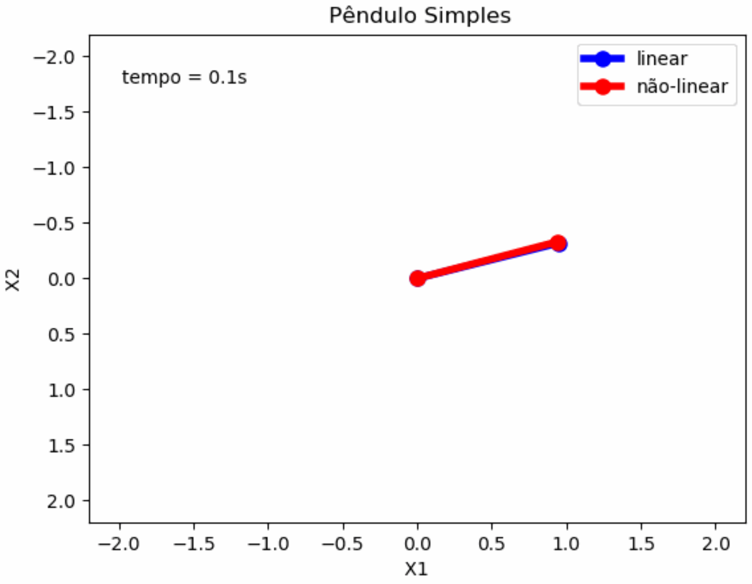
### VIBRAÇÕES LIVRES SEM AMORTECIMENTO

Ângulo Inicial: $\theta_0 = -120\,°$

Velocidade Inicial: $\dot{\theta}_0 = 200\,°/s$



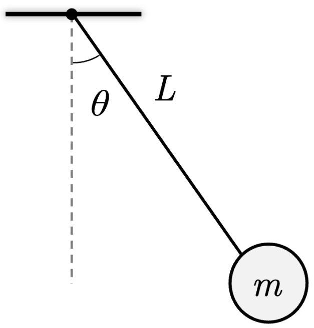



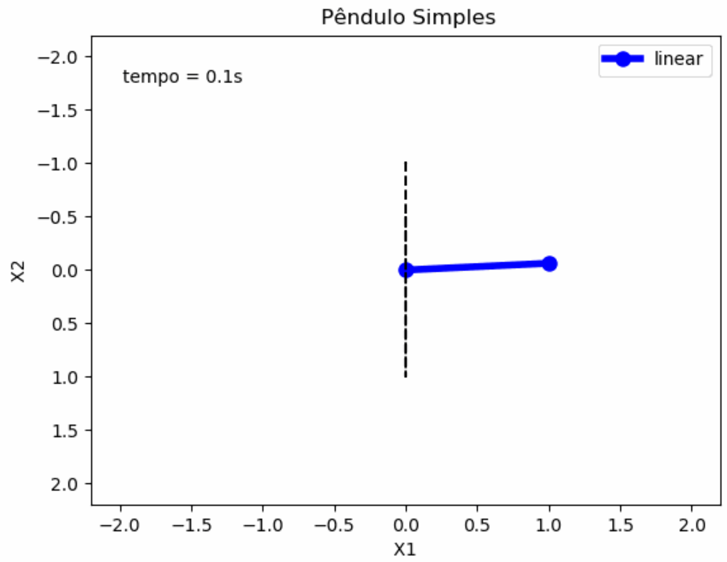
# Sistemas Lineares vs Não Lineares (Exemplo)

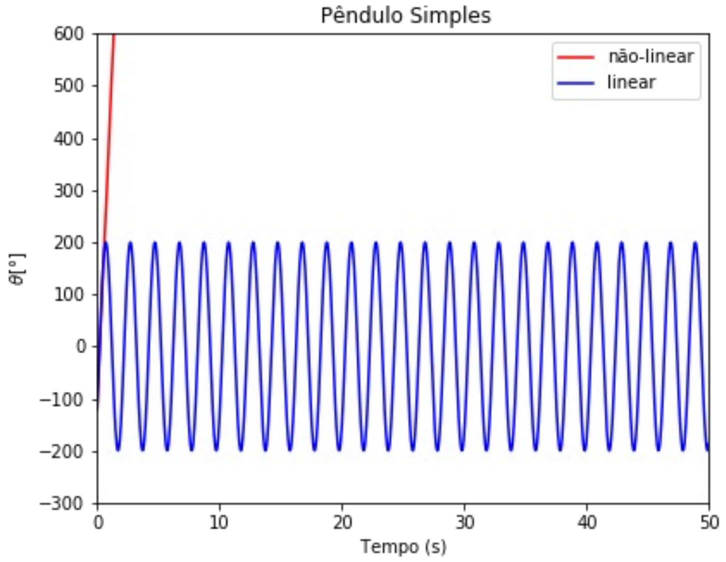
## VIBRAÇÕES LIVRES SEM AMORTECIMENTO

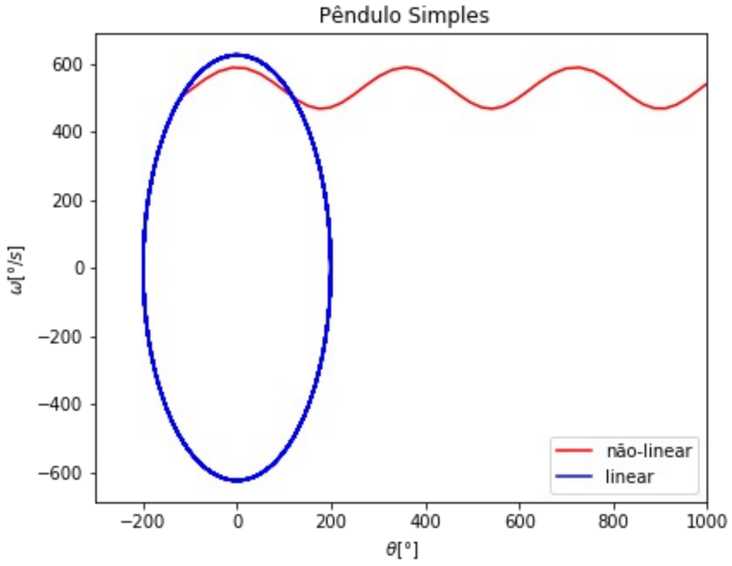
Ângulo Inicial: $\theta_0 = -120\,°$

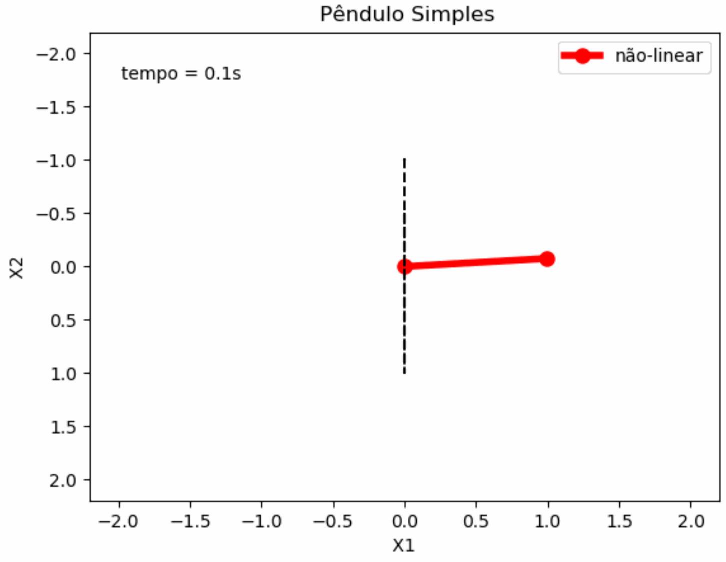
Velocidade Inicial: $\dot{\theta}_0 = 500\,°/s$



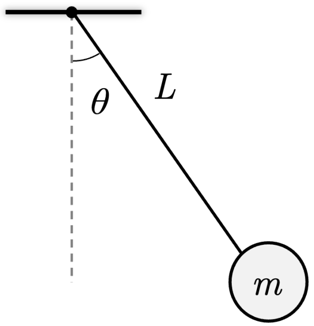
# Sistemas Lineares vs Não Lineares (Exemplo)

**VIBRAÇÕES LIVRES AMORTECIDAS**

Ângulo Inicial: $\theta_0 = -10\,°$

Velocidade Inicial: $\dot{\theta}_0 = 0\,°/s$

Dissipação: $2\zeta\omega_n = 0.08$

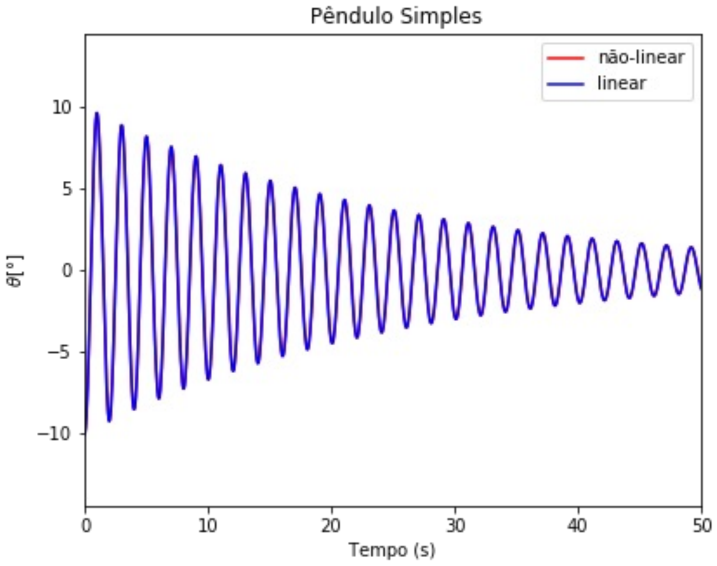


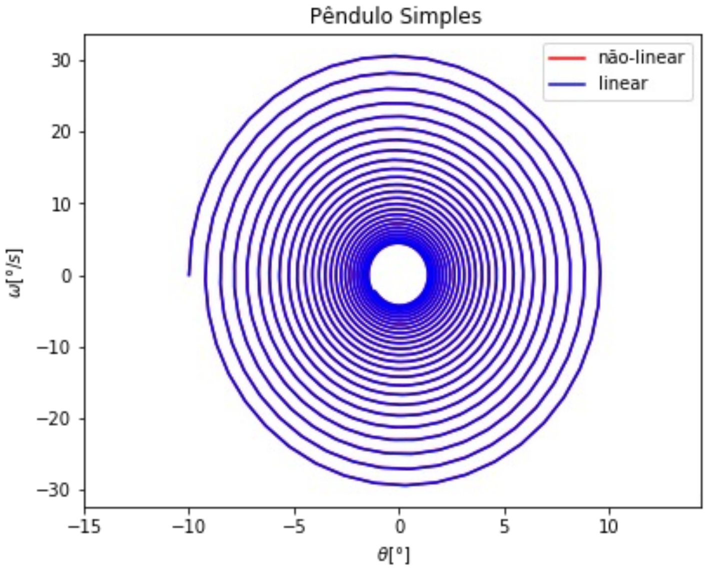

**Não Linear**

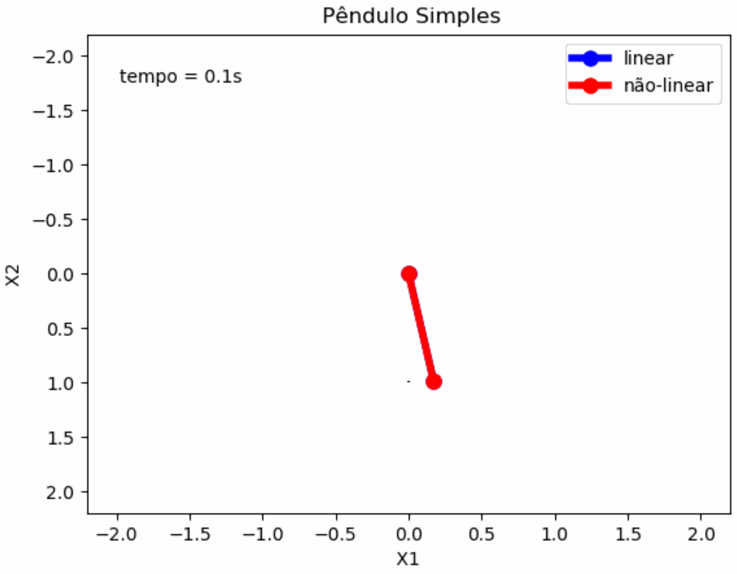
$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2 \sin(\theta) = f(t)}$$

**Linear**

$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = f(t)}$$



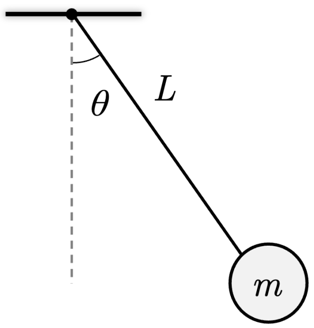
# Sistemas Lineares vs Não Lineares (Exemplo)

**VIBRAÇÕES LIVRES AMORTECIDAS**

Ângulo Inicial: $\theta_0 = -120\,°$

Velocidade Inicial: $\dot{\theta}_0 = 0\,°/s$

Dissipação: $2\zeta\omega_n = 0.08$

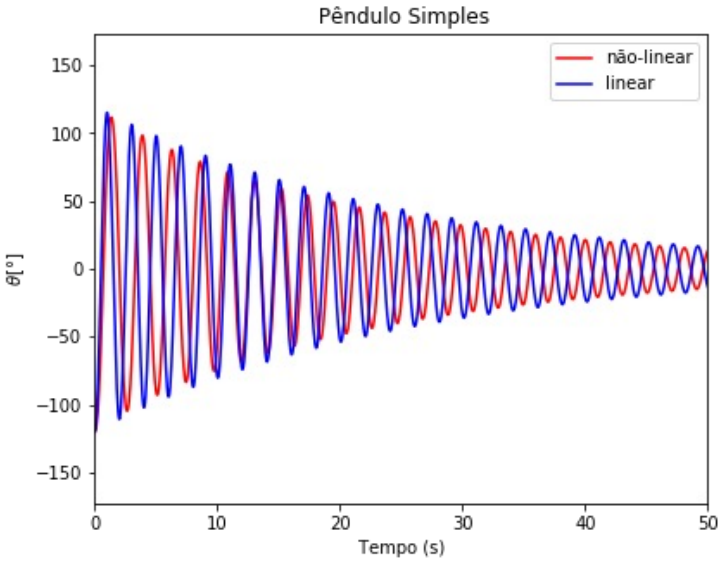


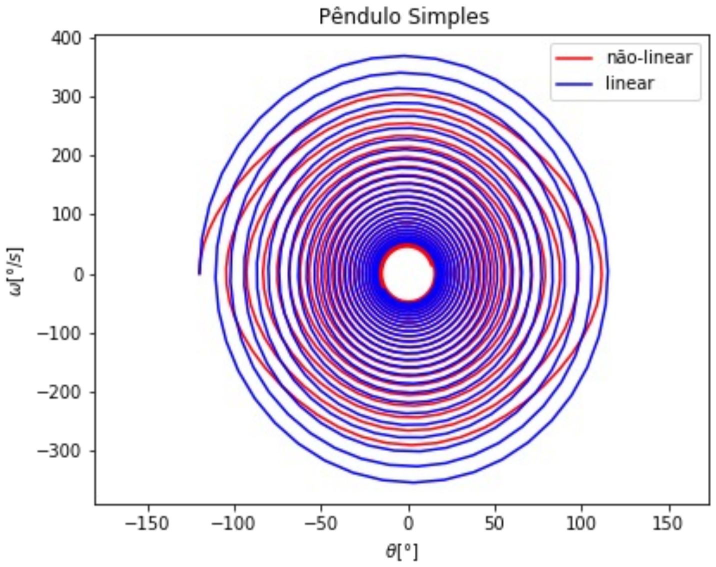

**Não Linear**

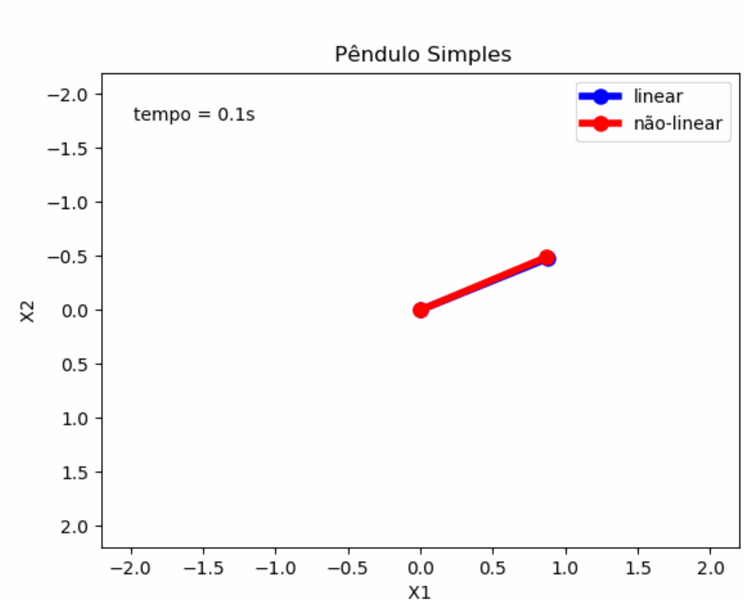
$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2 \sin(\theta) = f(t)}$$

**Linear**

$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = f(t)}$$



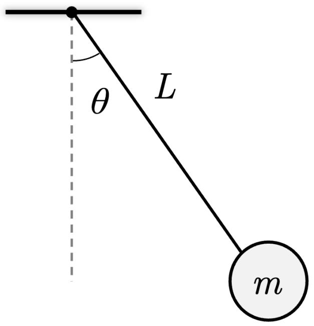
# Sistemas Lineares vs Não Lineares (Exemplo)

**VIBRAÇÕES LIVRES AMORTECIDAS**

Ângulo Inicial: $\theta_0 = -120\,°$

Velocidade Inicial: $\dot{\theta}_0 = 500\,°/s$

Dissipação: $2\zeta\omega_n = 0.2$

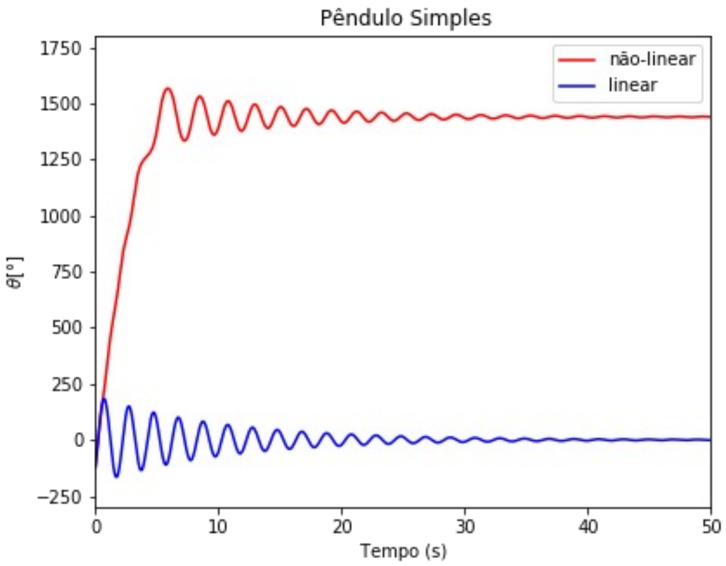


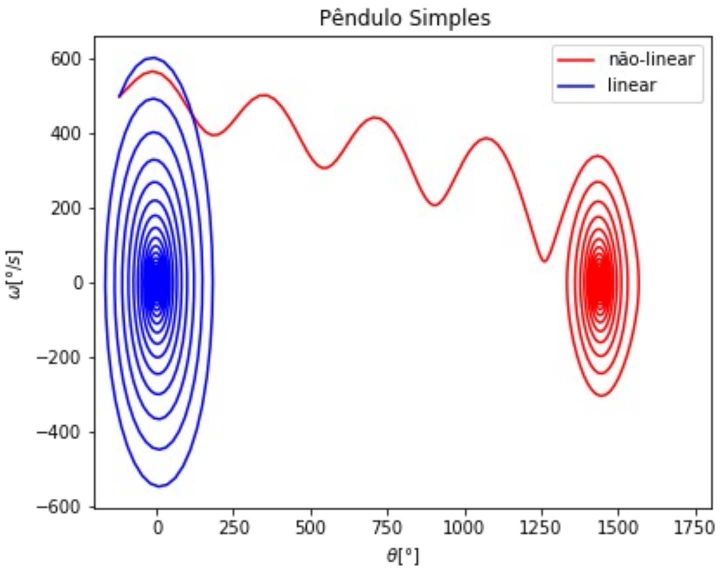

**Não Linear**

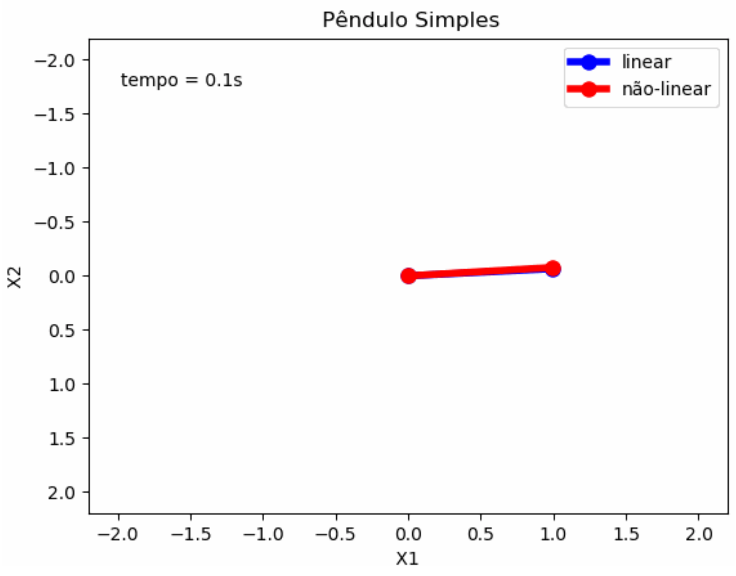
$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2 \sin(\theta) = f(t)}$$

**Linear**

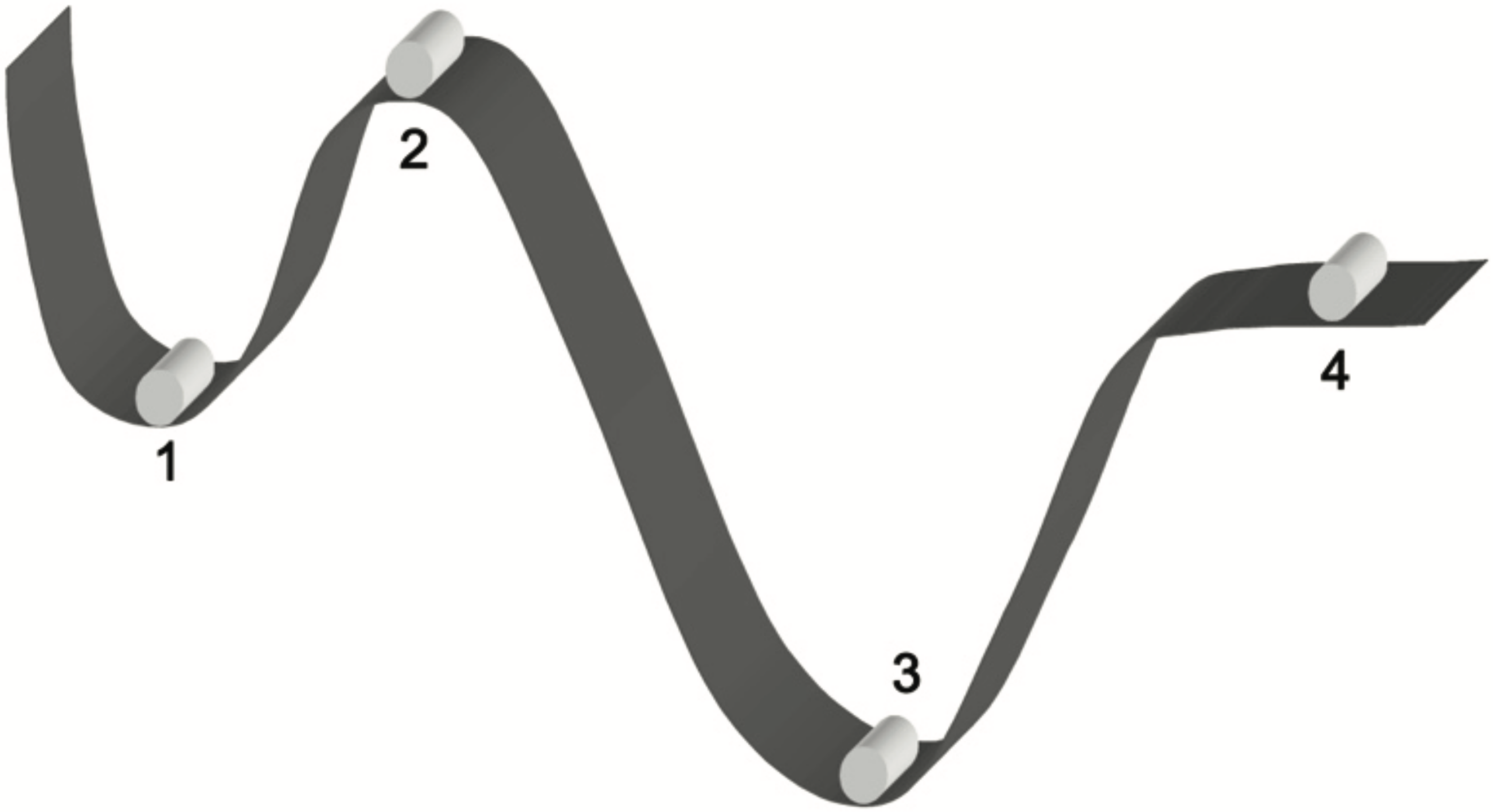
$$\boxed{\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = f(t)}$$



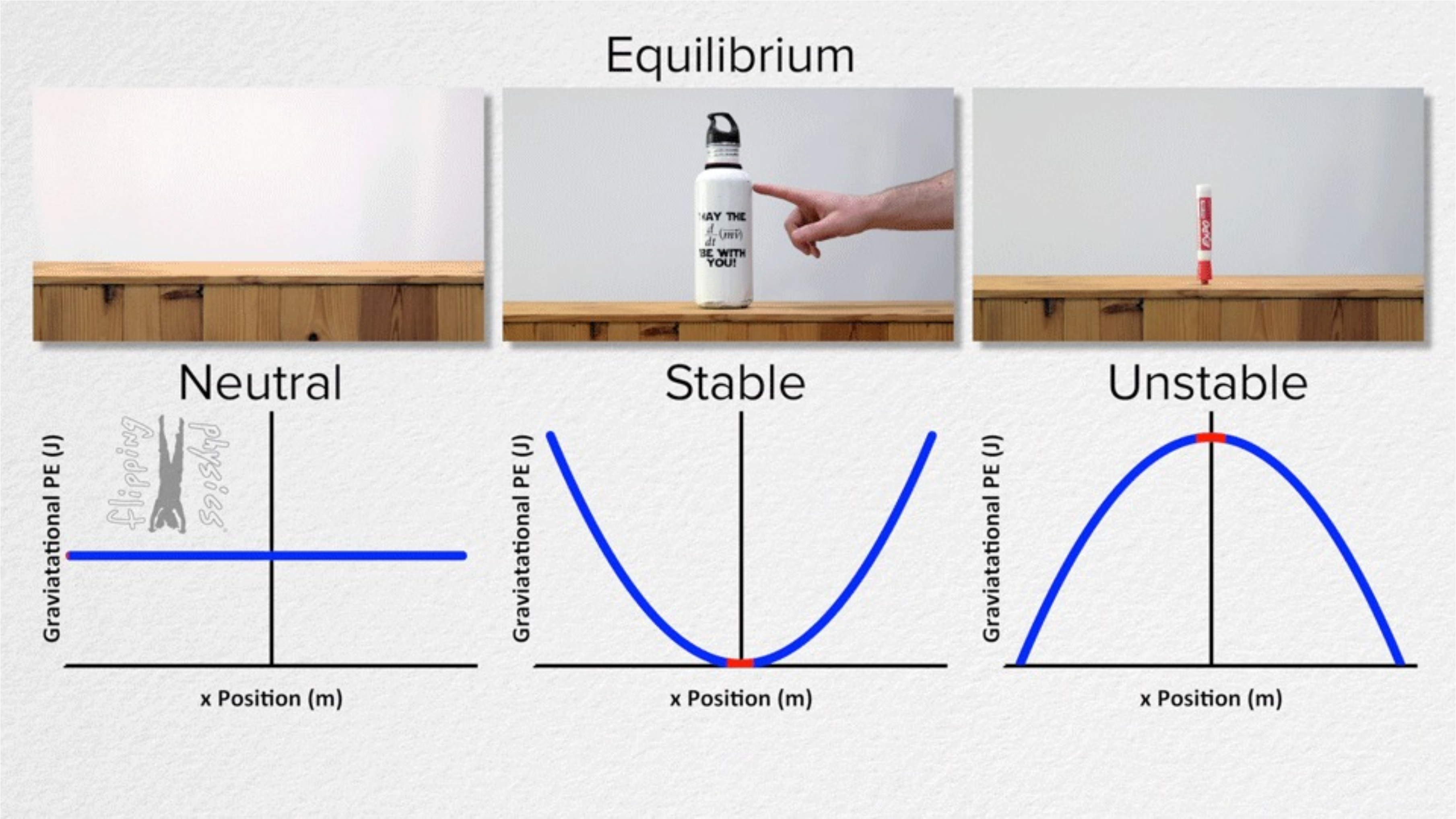



# Estabilidade



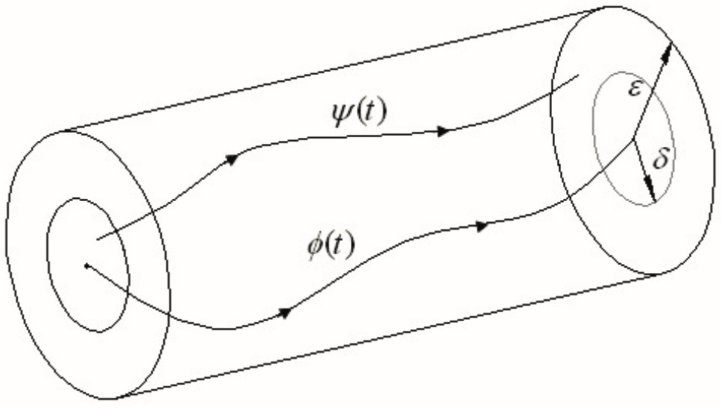
# Estabilidade e Equilíbrio de um Sistema Dinâmico



# Estabilidade de Lyapunov

**Aleksandr Lyapunov**

**(1857 – 1918)**

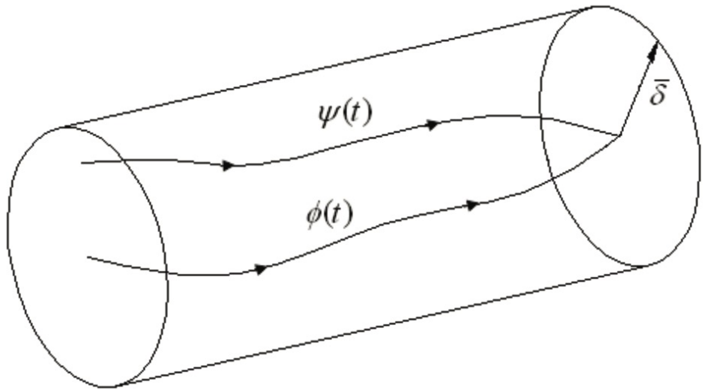



Solução Lyapunov estável

$$\delta = \delta(\varepsilon) > 0, \text{tal que}$$

$$|\psi(t_0) - \phi(t_0)| < \delta$$

$$|\psi(t) - \phi(t)| < \varepsilon$$




Solução assintoticamente estável

$$\bar{\delta} > 0, \text{tal que}$$

$$|\psi(t_0) - \phi(t_0)| < \bar{\delta}$$

$$\lim_{t \to \infty} |\psi(t) - \phi(t)| = 0$$

# Linearização

→ **Ferramenta importante para análise de um sistema localmente.**

$$\mathbf{x}(t) = \boldsymbol{\phi}(t) + \boldsymbol{\eta}; \qquad \mathbf{x}, \boldsymbol{\phi}, \boldsymbol{\eta} \in \mathbb{R}^n \qquad \text{(Solução Perturbada)}$$

→ **Podemos expandir $f(x)$ em série de Taylor até o primeiro termo:**

$$\mathbf{f}\big(\mathbf{x}(t)\big) = \mathbf{f}(\boldsymbol{\phi}(t) + \boldsymbol{\eta}) = \mathbf{f}\big(\boldsymbol{\phi}(t)\big) + \mathbf{J}\left(\mathbf{f}\big(\boldsymbol{\phi}(t)\big)\right)\Big|_{\boldsymbol{\eta}=\mathbf{0}} \cdot \boldsymbol{\eta} + \cdots$$

**Matriz Jacobiana**

$$\mathbf{J}\Big(\mathbf{f}\big(\boldsymbol{\phi}(t)\big)\Big) = \frac{\partial f_i\big(\boldsymbol{\phi}(t)\big)}{\partial x_j}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$$

→ **Portanto, expandindo para equação de evolução:**

$$\dot{\mathbf{x}}(t) = \dot{\boldsymbol{\phi}}(t) + \dot{\boldsymbol{\eta}} = \mathbf{f}\big(\mathbf{x}(t)\big) = \mathbf{f}(\boldsymbol{\phi}(t) + \boldsymbol{\eta}) = \mathbf{f}\big(\boldsymbol{\phi}(t)\big) + \mathbf{J}\left(\mathbf{f}\big(\boldsymbol{\phi}(t)\big)\right)\Big|_{\boldsymbol{\eta}=\mathbf{0}} \cdot \boldsymbol{\eta} + \cdots$$

→ **Como $\dot{\boldsymbol{\phi}}(t) = \mathbf{f}\big(\boldsymbol{\phi}(t)\big)$, então:**

$$\boxed{\dot{\boldsymbol{\eta}} = \mathbf{J}\left(\mathbf{f}\big(\boldsymbol{\phi}(t)\big)\right)\Big|_{\boldsymbol{\eta}=\mathbf{0}} \cdot \boldsymbol{\eta}}$$

(Sistema Linearizado em torno da solução)

→ **A linearização é feita em torno de algo (uma perturbação), portanto é sempre relativa;**

→ **A estabilidade do sistema linearizado corresponde ao sistema não linear**

# Equilíbrio de um Sistema Dinâmico

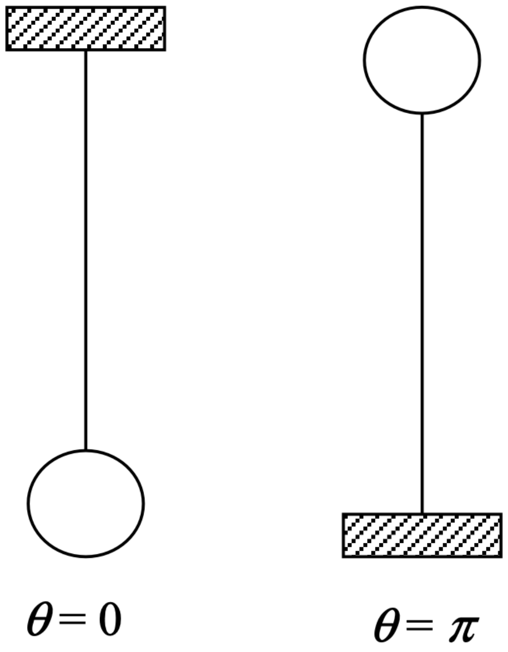
→ **Um ponto de equiíbrio se refere a pontos nos quais o sistema se mantém estacionário à medida que o tempo evolui (não varia com o tempo).**

$$\dot{\mathbf{x}}(t) = \mathbf{0}; \qquad \mathbf{x} \in \mathbb{R}^n$$

→ **Exemplo: Pêndulo simples**

$$\dot{\mathbf{x}}(t) = \begin{Bmatrix} \dot{\theta}(t) \\ \ddot{\theta}(t) \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}(t) \\ -2\zeta\omega_n\dot{\theta}(t) - \omega_n^2 \sin\big(\theta(t)\big) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left\{\bar{\theta}, \dot{\bar{\theta}}\right\} = \{0, n\pi\}$$



# Natureza do Equilíbrio de um Sistema Dinâmico

→ Aplica-se uma perturbação no ponto de equilíbrio para avaliar se ele diverge ou converge (analisa-se localmente)

$$\boxed{\dot{\boldsymbol{\eta}} = \mathbf{J}\big(\mathbf{f}(\bar{\mathbf{x}})\big)\Big|_{\boldsymbol{\eta}=\mathbf{0}} \cdot \boldsymbol{\eta}}$$

→ A solução desse sistema linear é dada pela exponencial da matrix:

$$\boldsymbol{\eta}(t) = e^{t\mathbf{J}\left(\mathbf{f}(\bar{\mathbf{x}})\right)}\boldsymbol{\eta}(0)$$

→ O termo exponencial depende dos autovalores da matrix Jacobiana:

$$(\mathbf{J} - \lambda\mathbf{I}) = \mathbf{0}$$

$$\lambda = Re(\lambda) + i\, Im(\lambda)$$

$$\Rightarrow$$

1. $Estável\ se\ \{\lambda_i \in \mathbb{C} \mid Re(\lambda_i) < 0, \forall i\}$
2. $Instável\ se\ \{\lambda_i \in \mathbb{C} \mid Re(\lambda_i) > 0, \exists i\}$
3. $Centro\ se\ \{\lambda_i \in \mathbb{C} \mid Re(\lambda_i) = 0, \exists i\}$

→ A estabilidade do sistema linearizado

# Natureza do Equilíbrio de um Sistema Dinâmico

$$1.\, Estável\ se\ \{\lambda_i \in \mathbb{C} \mid Re(\lambda_i) < 0, \forall i\}$$

$$2.\, Instável\ se\ \{\lambda_i \in \mathbb{C} \mid Re(\lambda_i) > 0, \exists i\}$$

$$3.\, Centro\ se\ \{\lambda_i \in \mathbb{C} \mid Re(\lambda_i) = 0, \exists i\}$$

→ **<u>Teorema de Hartman-Grobman (Em linhas gerais)</u>: A estabilidade do sistema linearizado na vizinhança do ponto de equilíbrio corresponde ao sistema não linear desde que esse ponto seja hiperbólico.**

→ **Ponto Hiperbólico: ponto no qual a parte real é diferente de zero** $(Re(\lambda_i) \neq 0)$

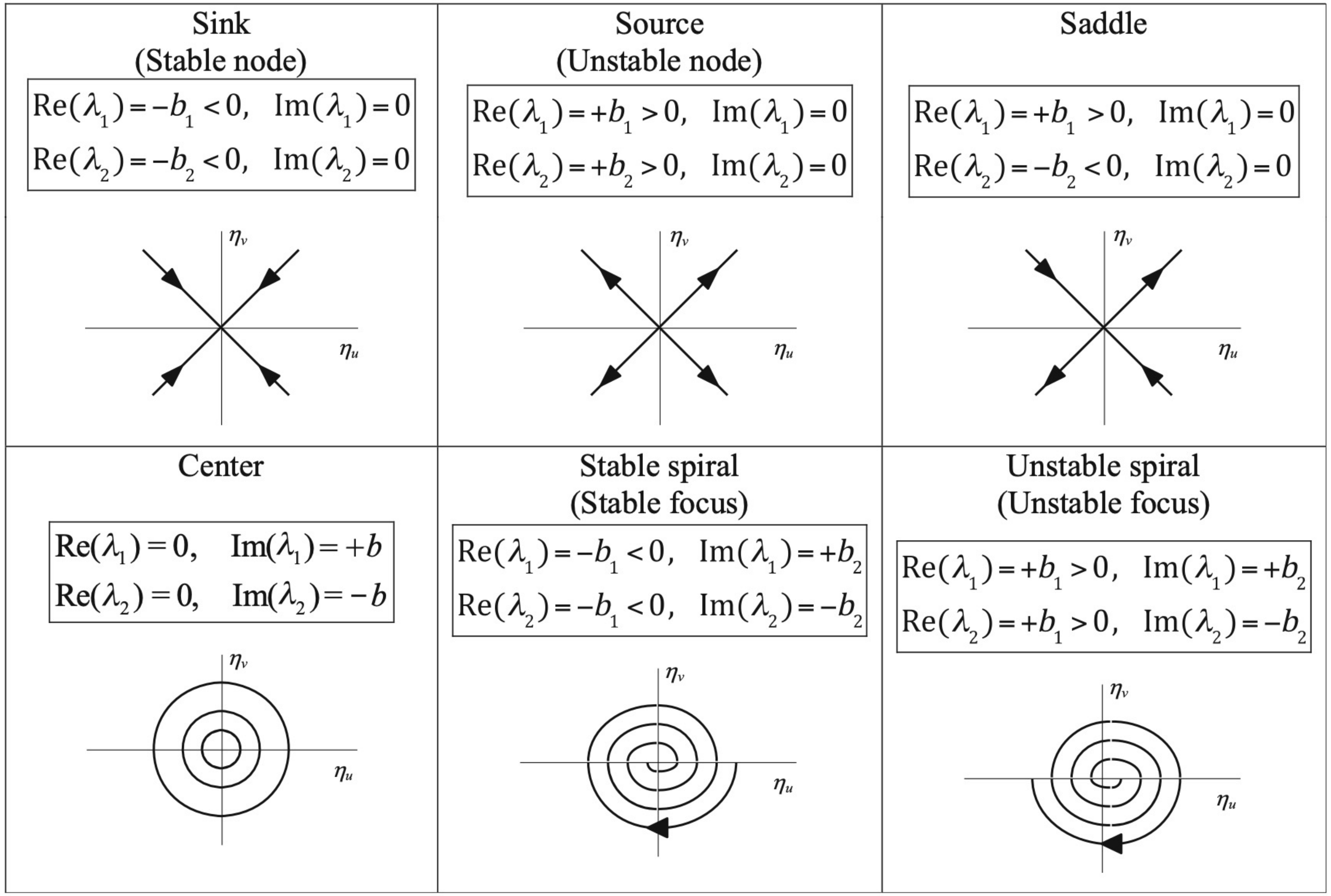
$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "h"'] & f(x) \arrow[d, "h"] \\
y \arrow[r, "g"'] & g(y)
\end{tikzcd}$$

**Homeorfismo**

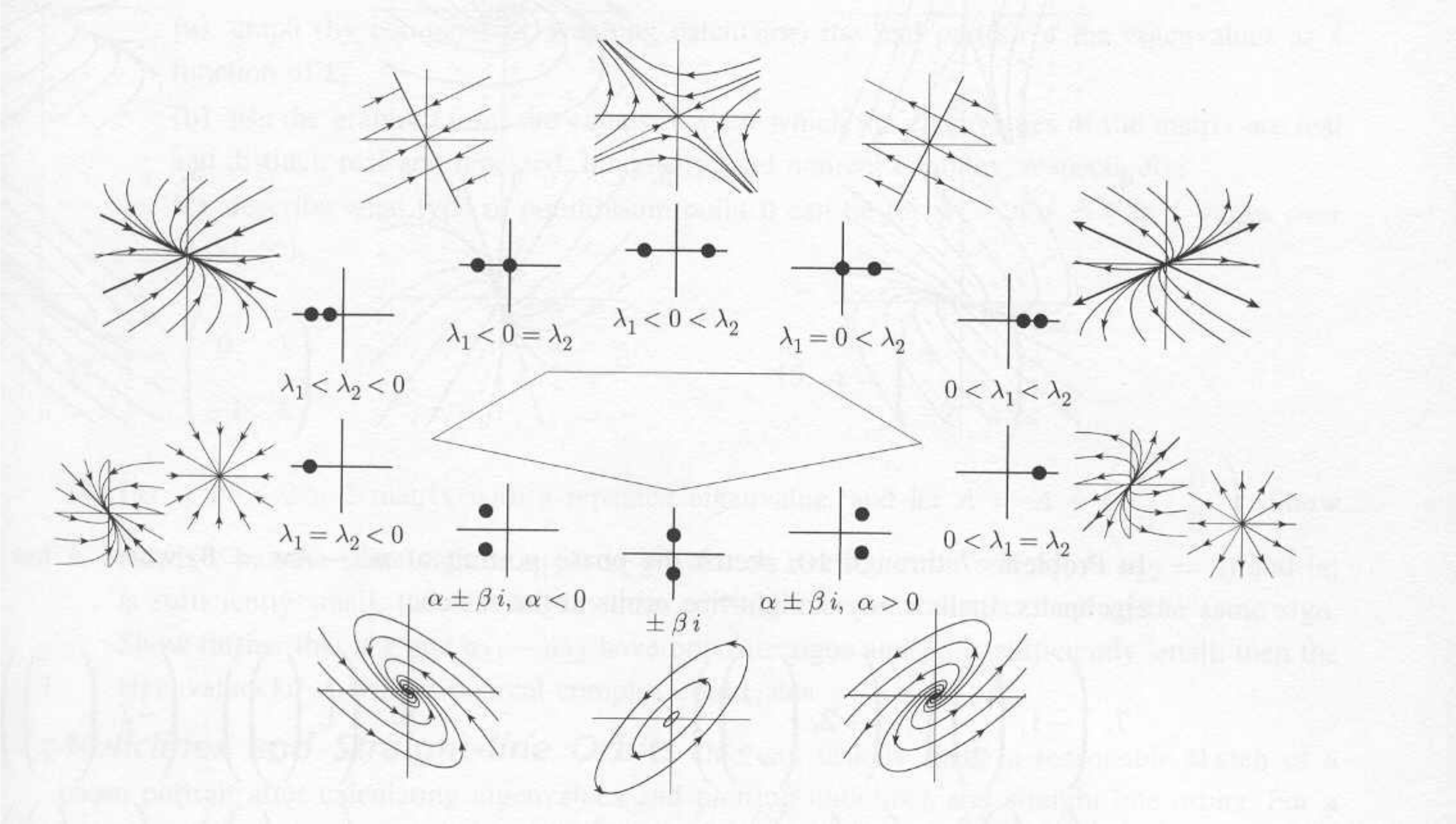
**(Equivalência Topológica)**

# Natureza do Equilíbrio de um Sistema Dinâmico

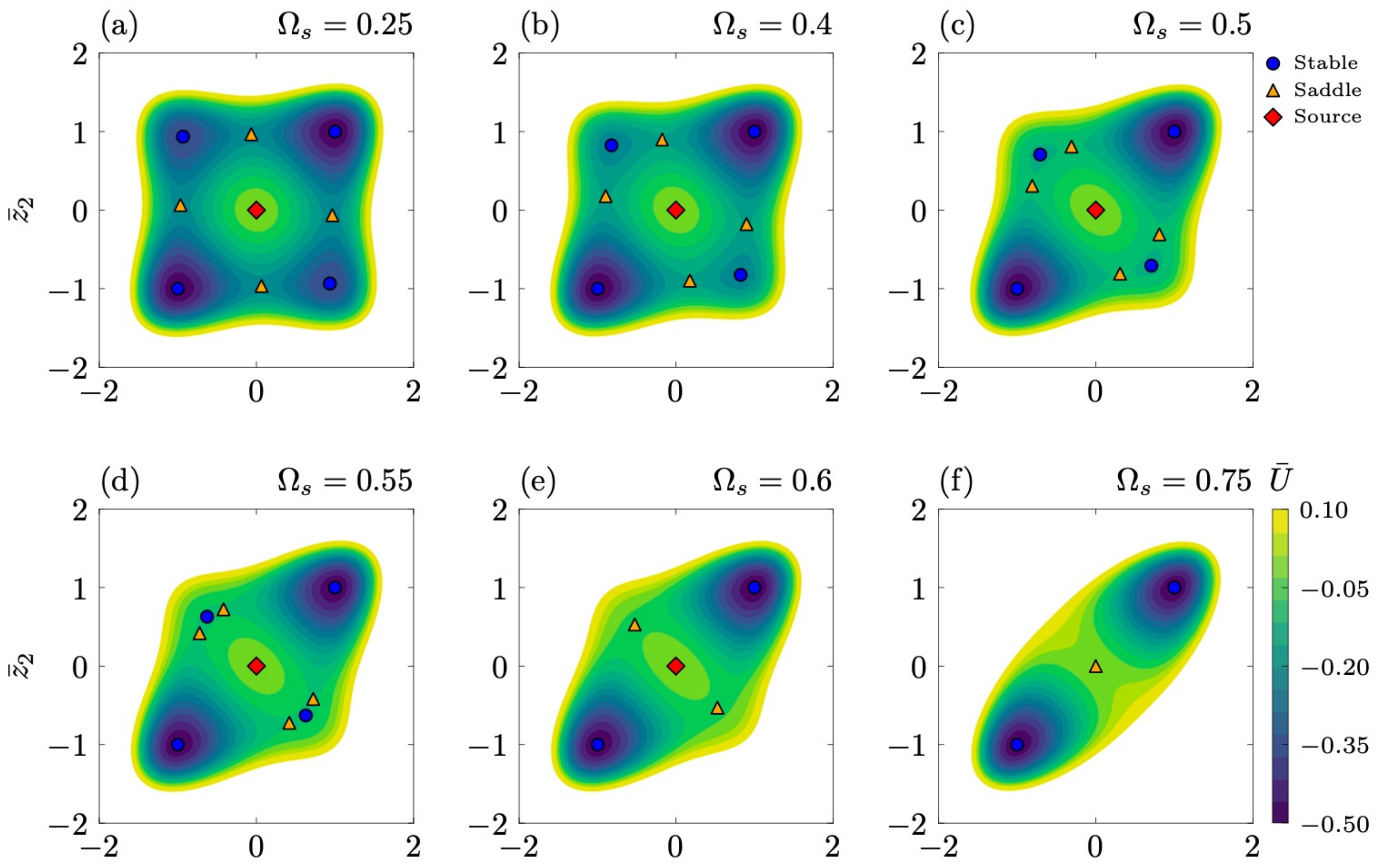
$$\dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t)\big); \qquad \mathbf{x}(t) \in \mathbb{R}^2$$



# Natureza do Equilíbrio de um Sistema Dinâmico

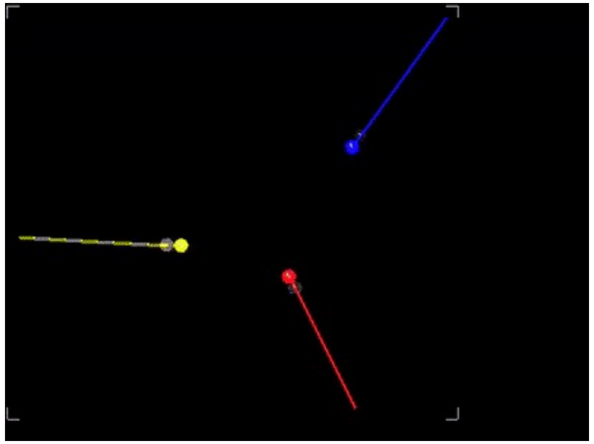


# Natureza do Equilíbrio de um Sistema Dinâmico

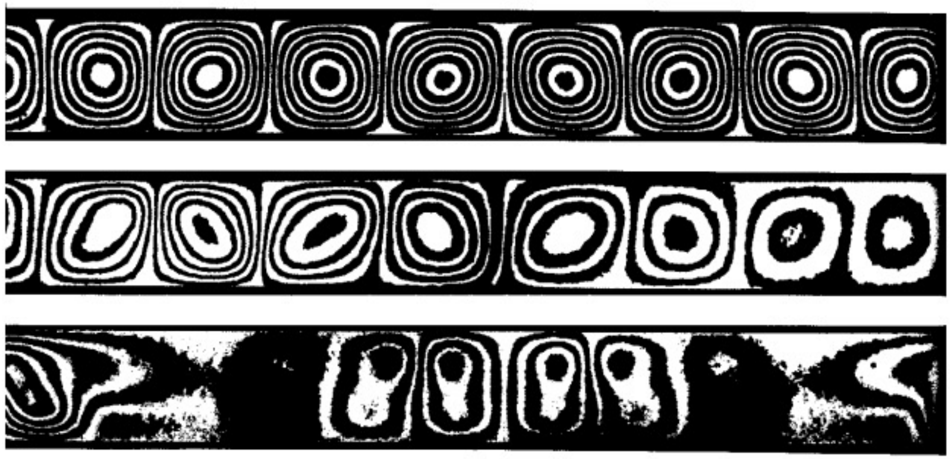


# Caos

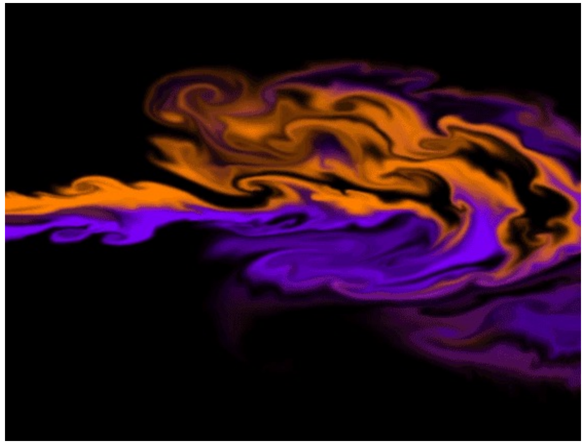
## Mecânica Celeste



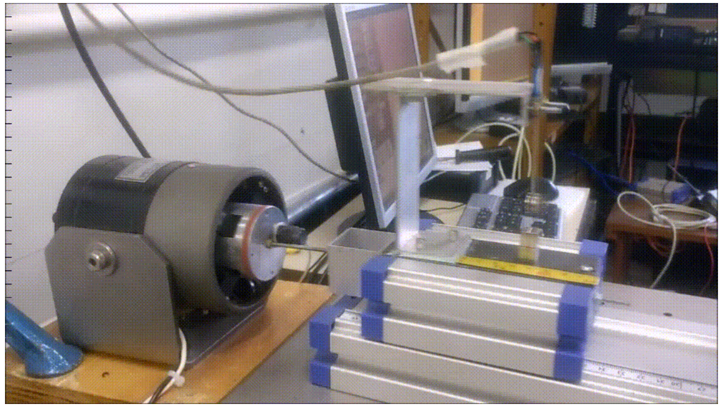
## Convecção Natural



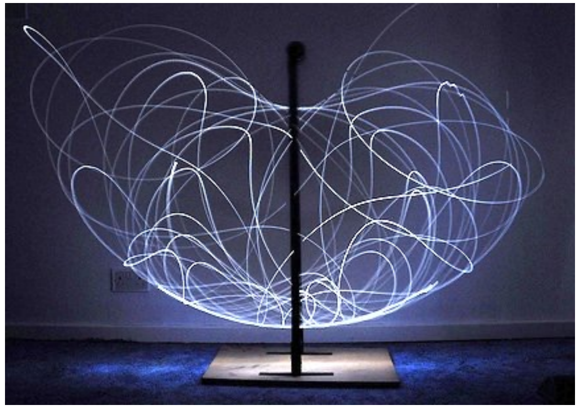
## Turbulência

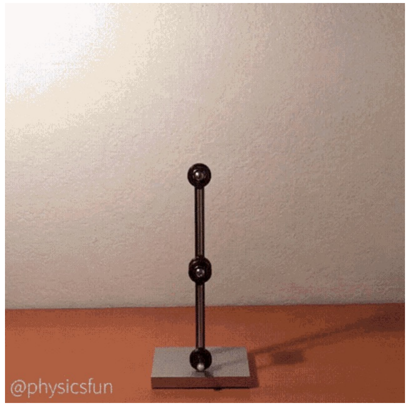


## Coletor de Energia Biestável

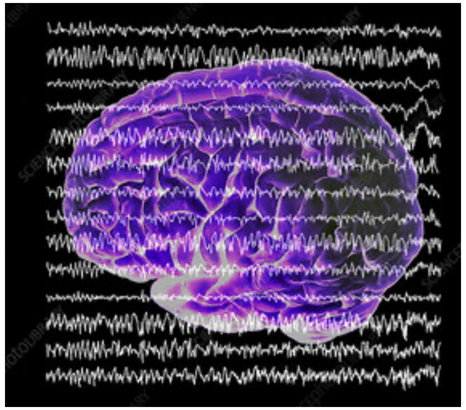


## Pêndulo Duplo





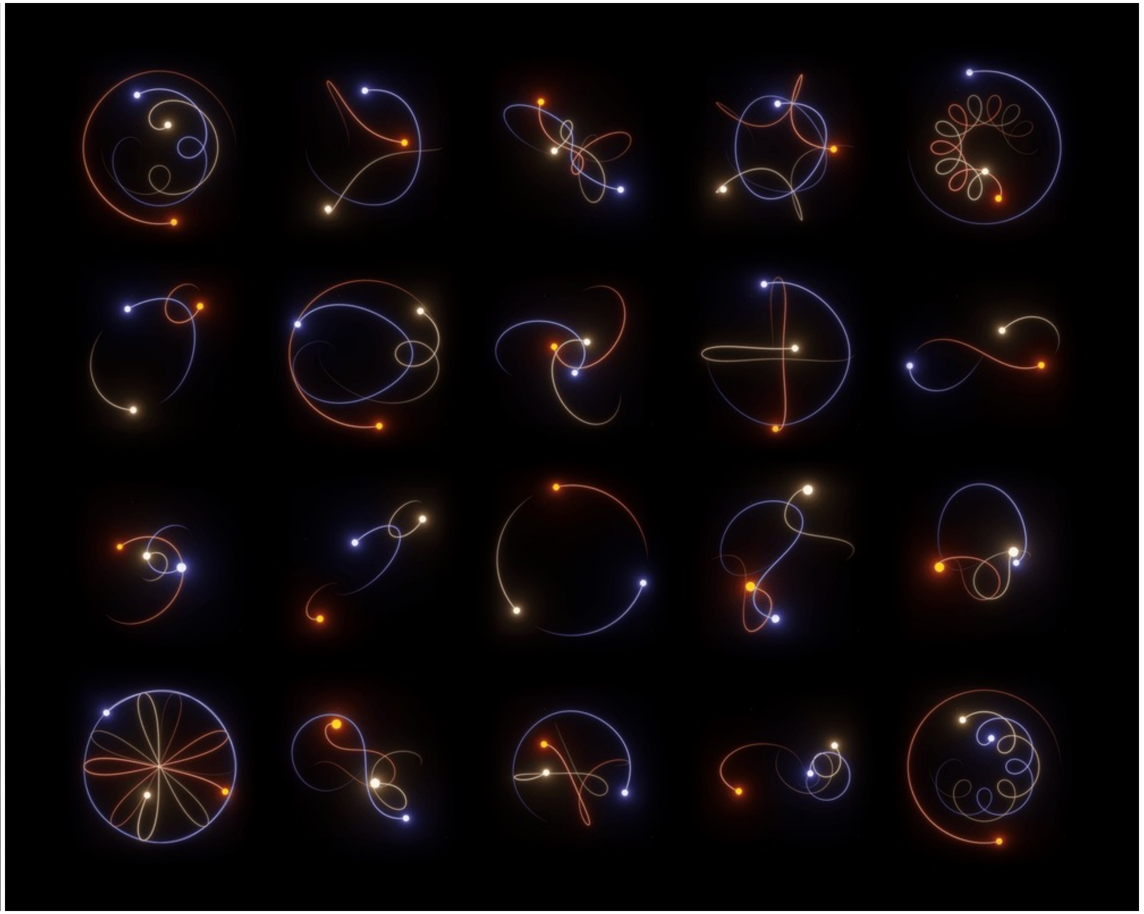

## Ondas Cerebrais



# Caos

**Henri Poincaré (1854 – 1912)**

**Problema dos 3 corpos (1885 – 1889)**

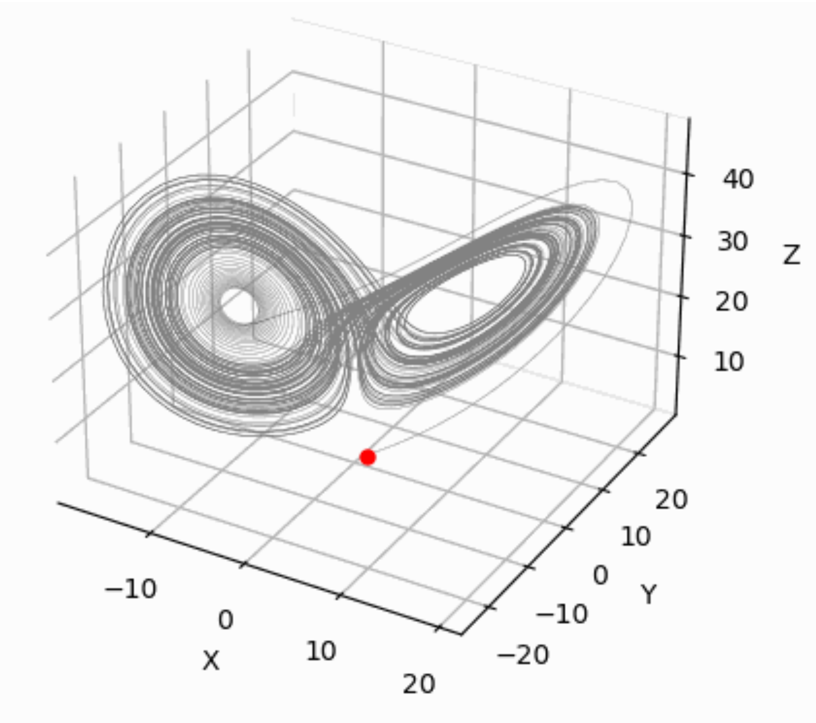


# Caos

**Edward Lorenz (1917 – 2008)**

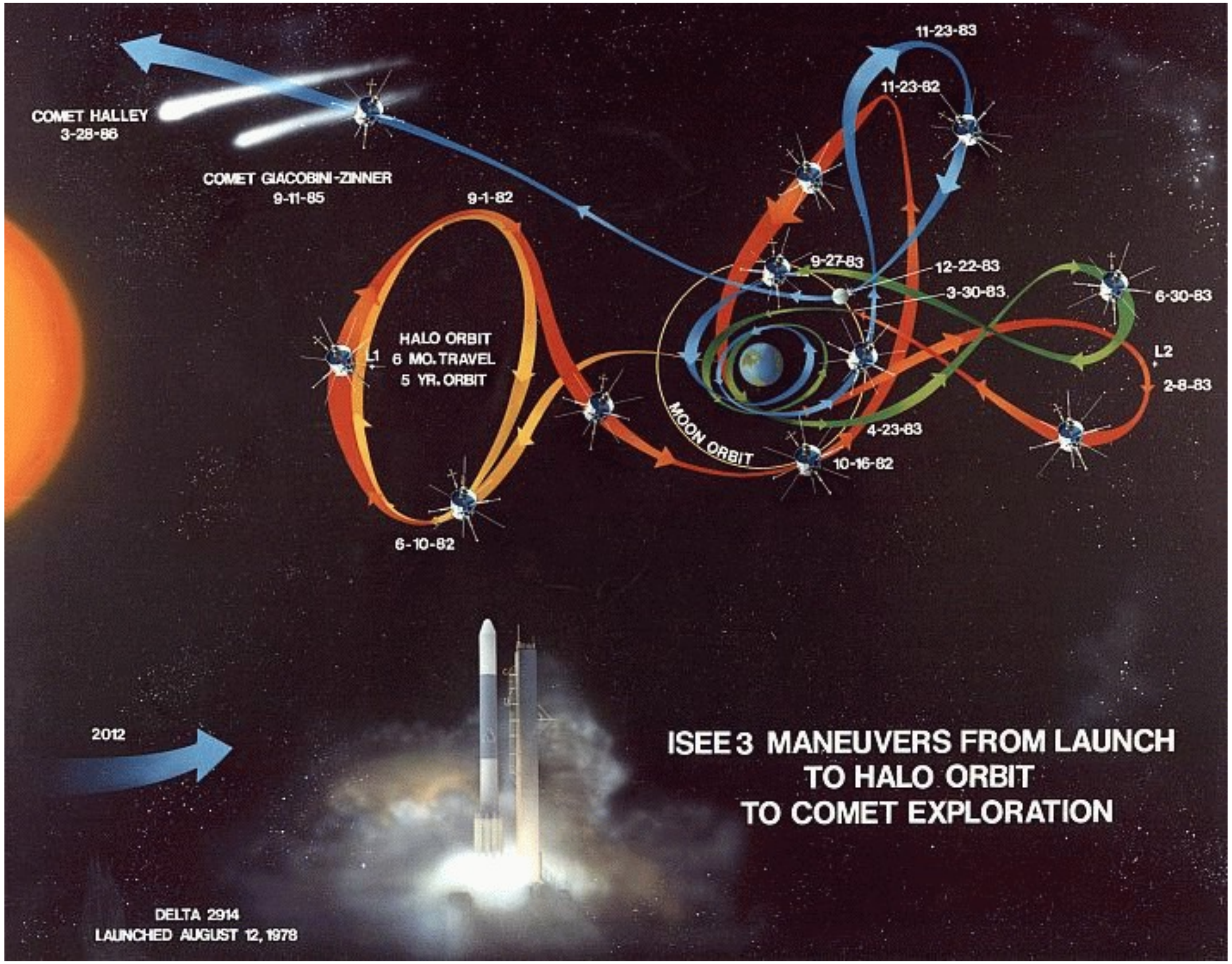
→ Abordou o problema computacionalmente

**Convecção Atmosférica Simplificada (1963)**

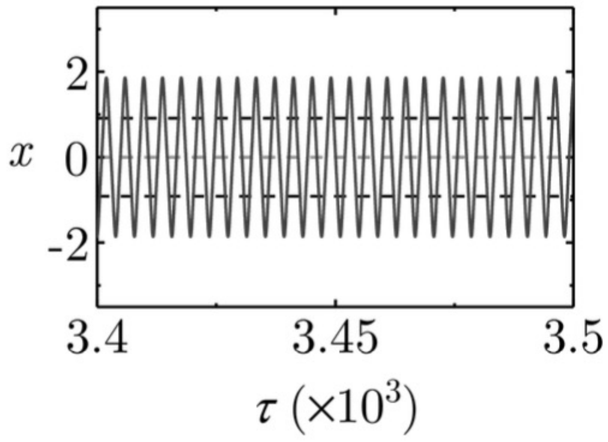



$$\dot{\mathbf{x}}(t) = \begin{Bmatrix} \sigma[y(t) - x(t)] \\ x(t)[\rho - z(t)] - y(t) \\ x(t)y(t) - \beta z(t) \end{Bmatrix}$$

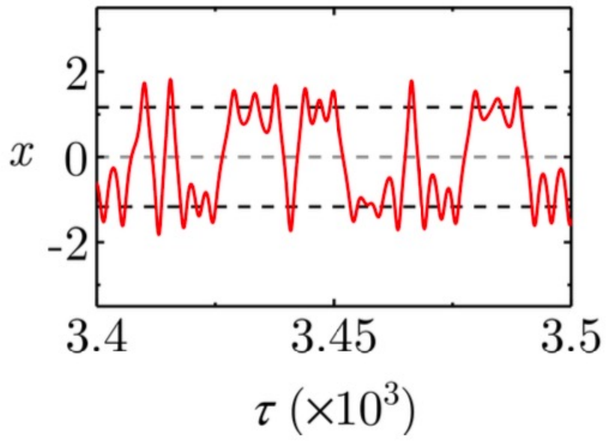
# Caos



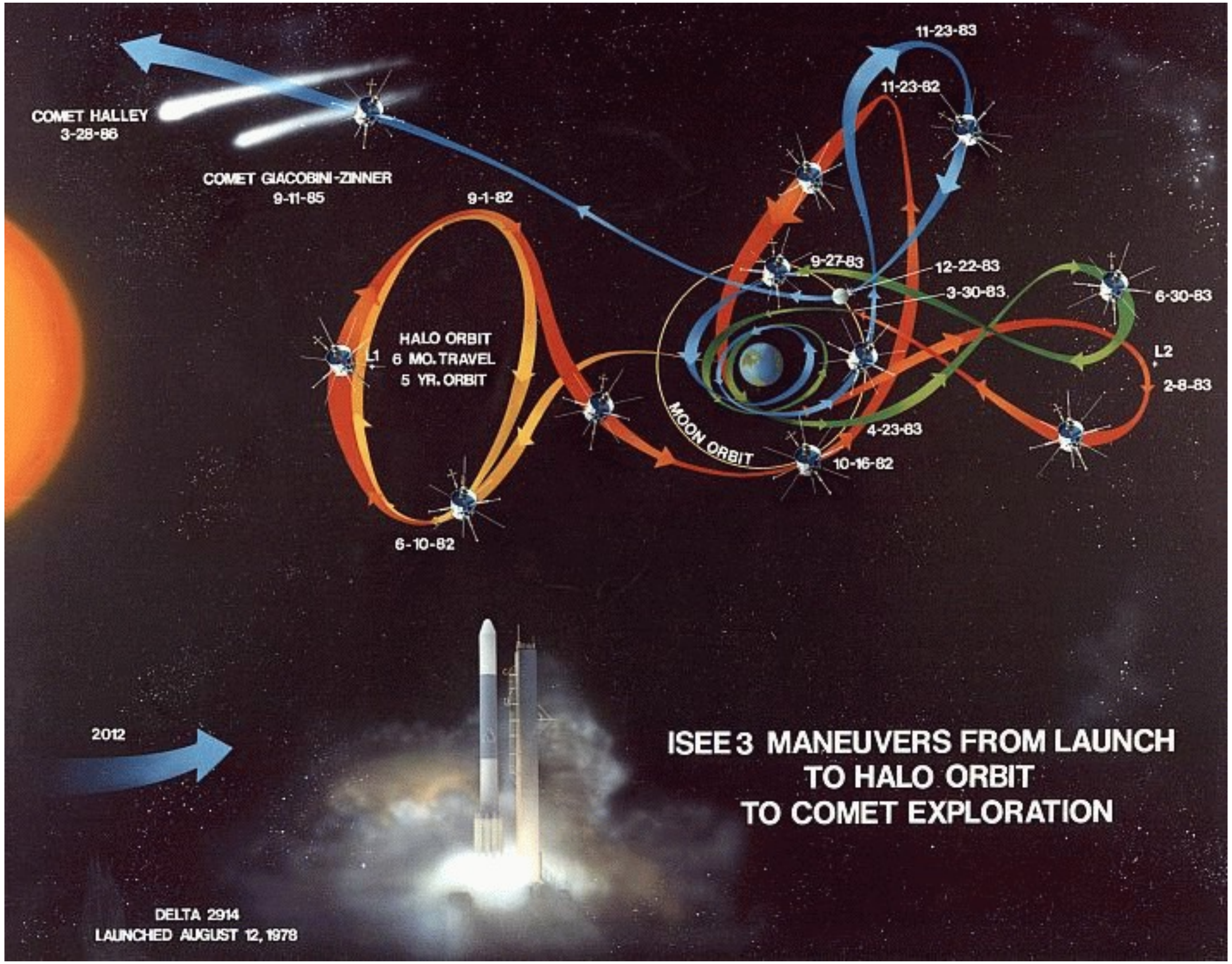

## Dinâmica Periódica




## Dinâmica Caótica



# Caos



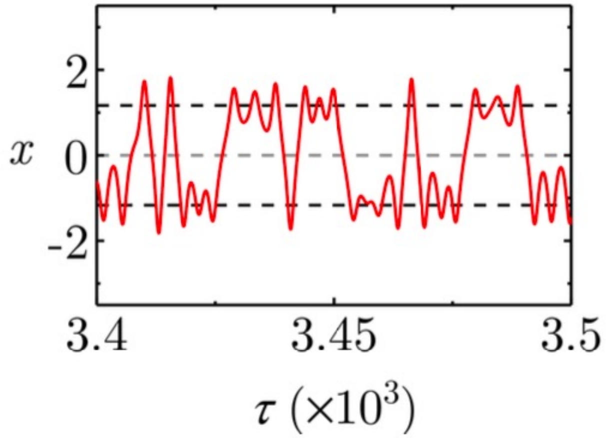

→ Sensibilidade à condições iniciais;

→ Instável localmente, estável globalmente;

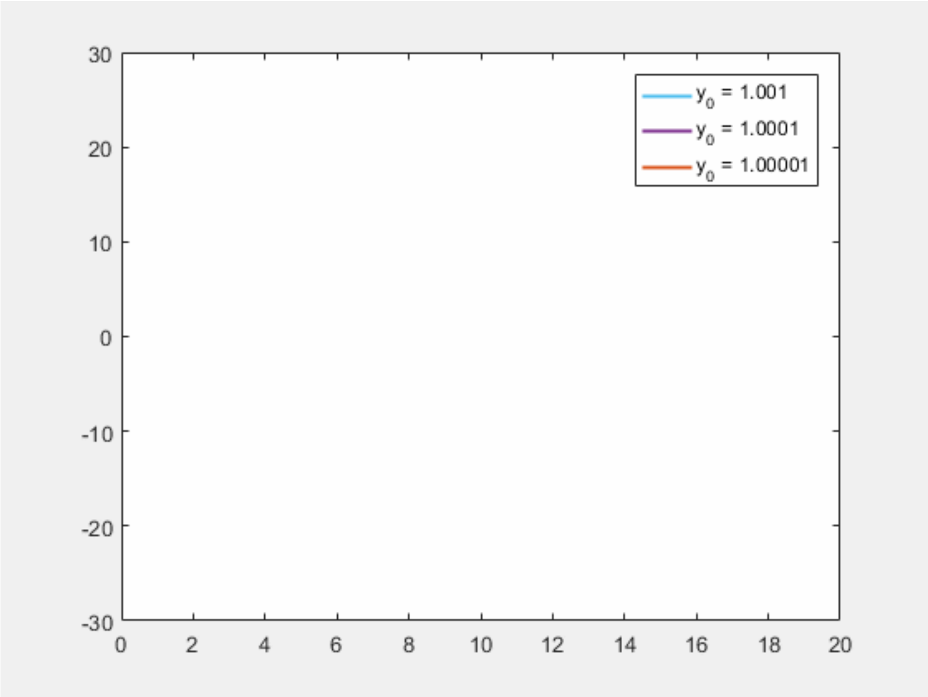
→ Determinístico;

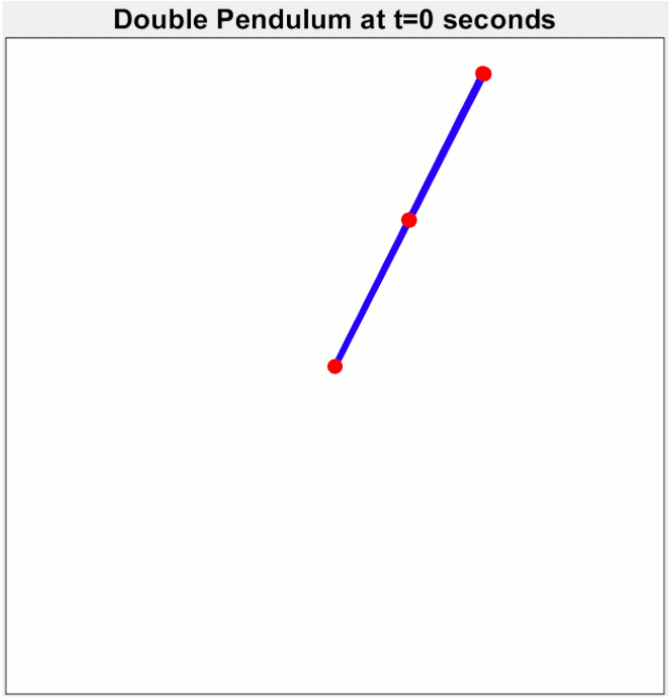
→ Praticamente imprevisível em períodos de tempo longos.

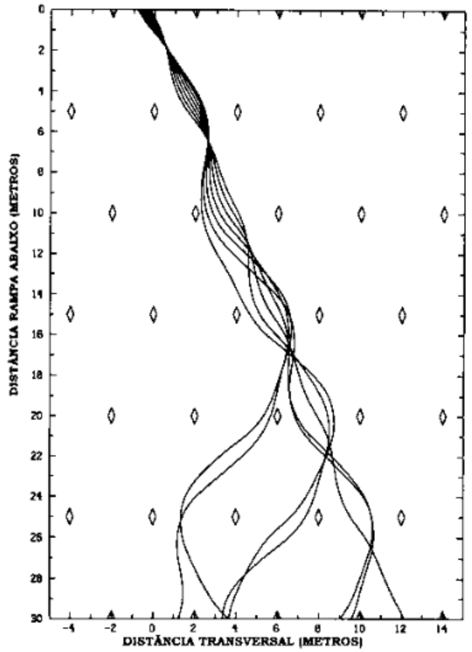
**Dinâmica Caótica**



# Caos

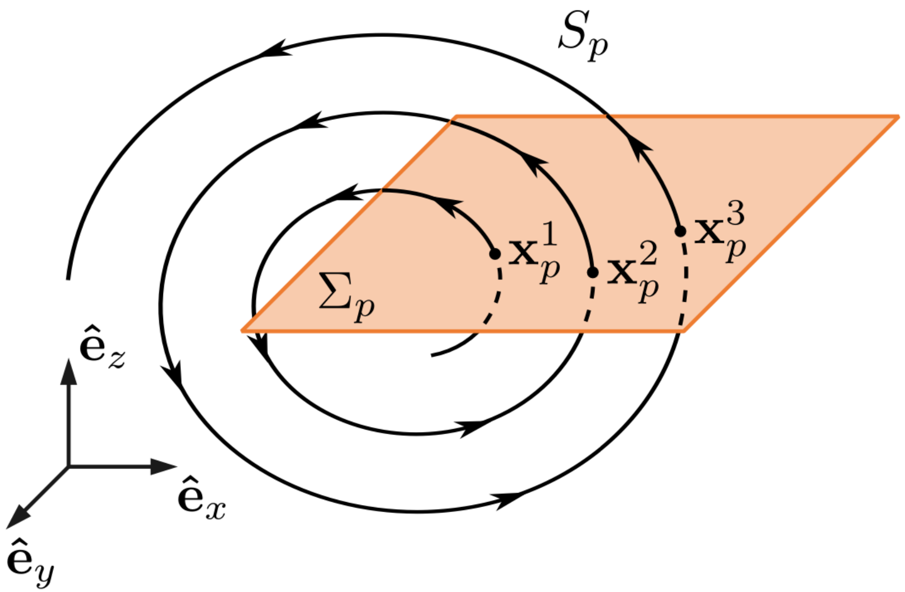

30
20
10
0
-10
-20
-30
0
2
4
6
8
10
12
14
16
18
20
$y_0$ = 1.001
$y_0$ = 1.0001
$y_0$ = 1.00001


Double Pendulum at t=0 seconds

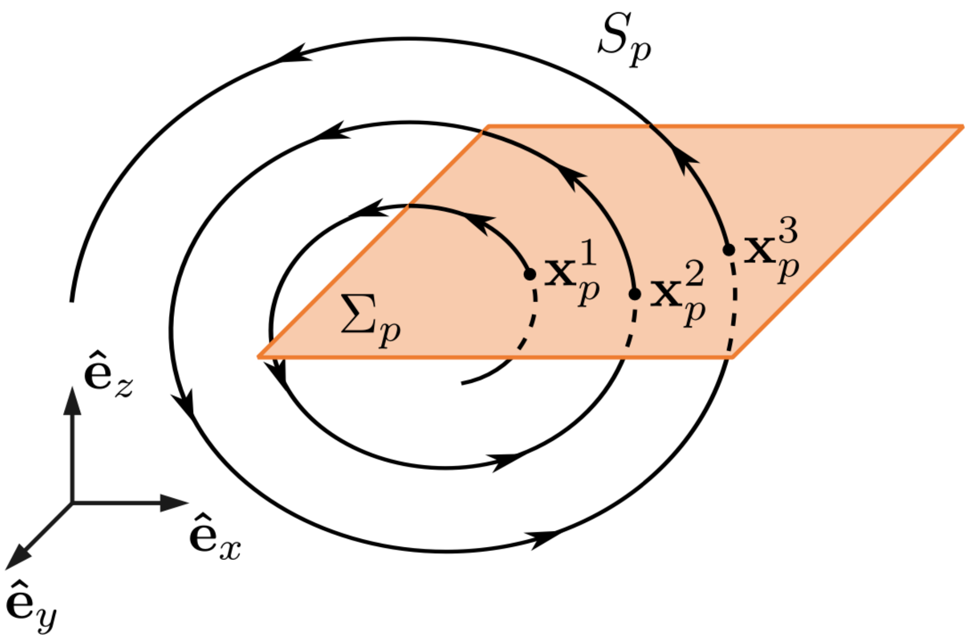

DISTÂNCIA RAMPA ABAIXO (METROS)
DISTÂNCIA TRANSVERSAL (METROS)

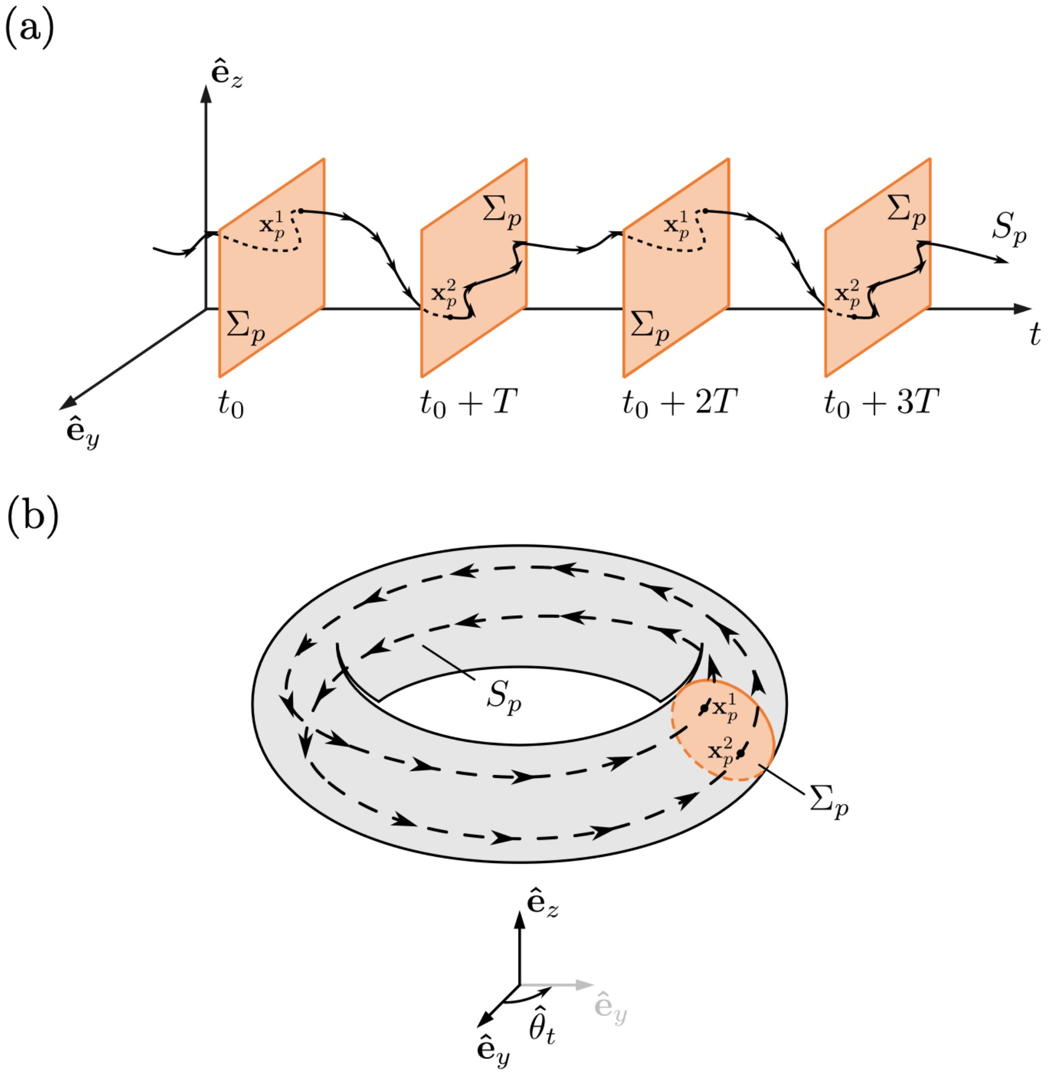
# Mapas de Poincaré

→ Movimento das hélices está sincronizado com a taxa de aquisição de imagens da câmera utilizada (Visão estroboscópica)

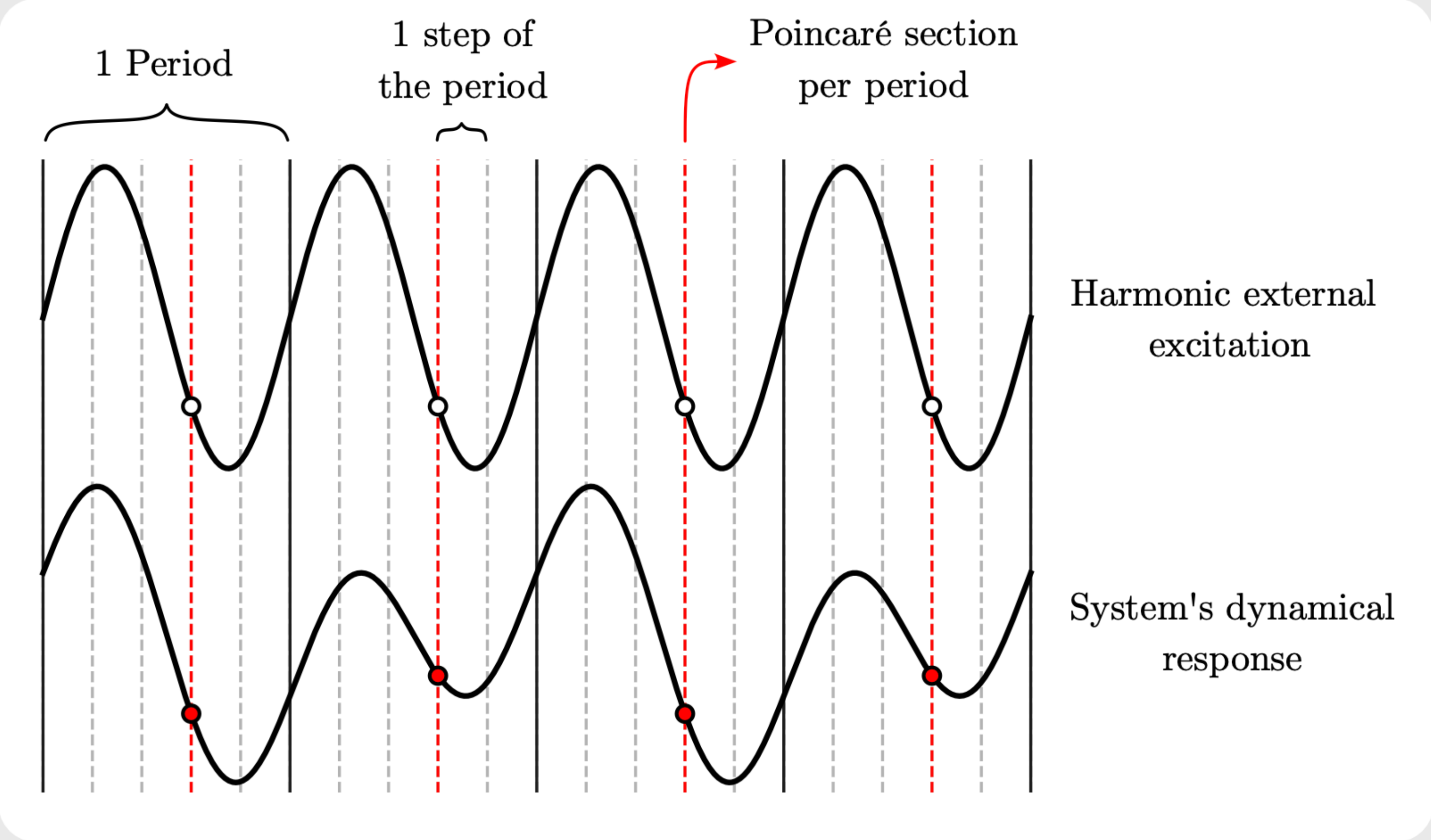


# Mapas de Poincaré

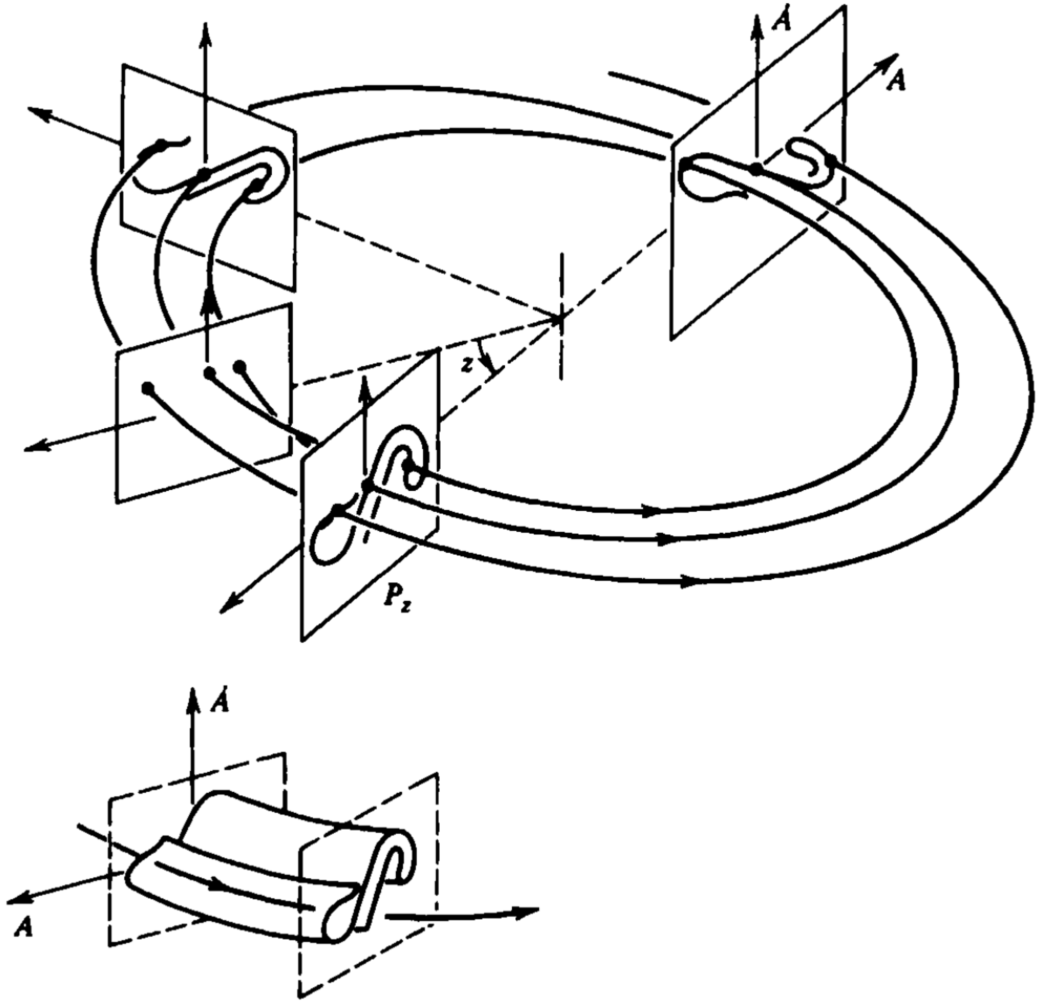


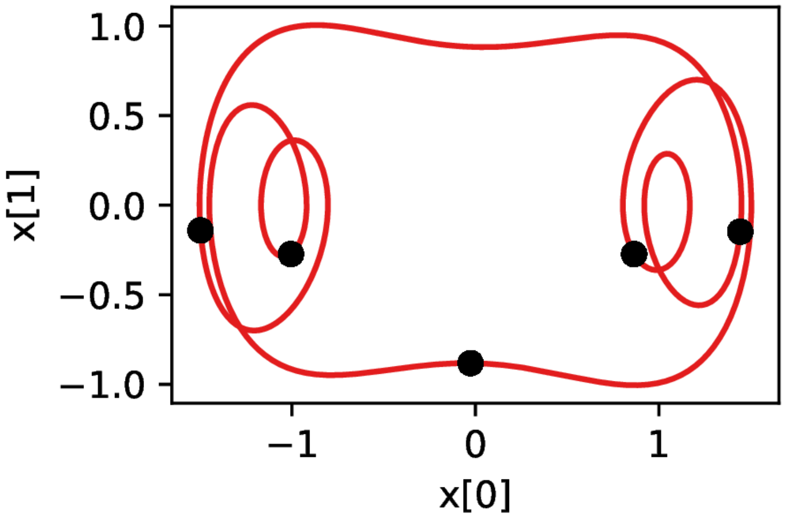



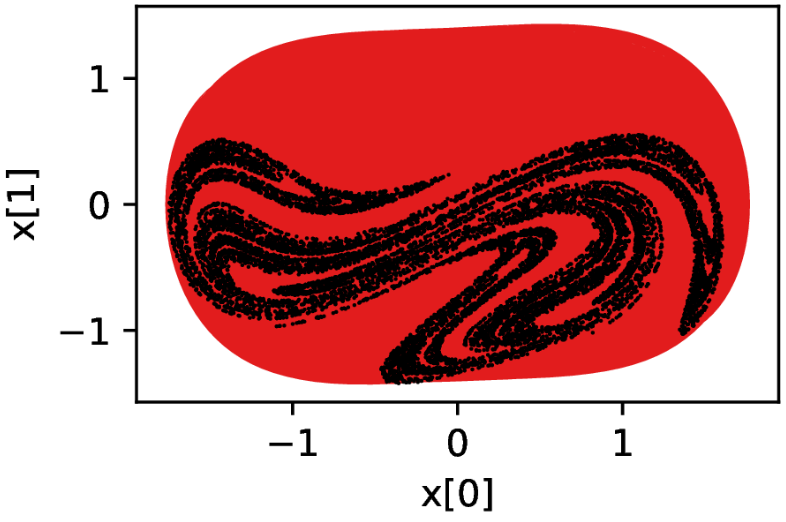
# Mapas de Poincaré (Sistemas com Excitação Harmônica)



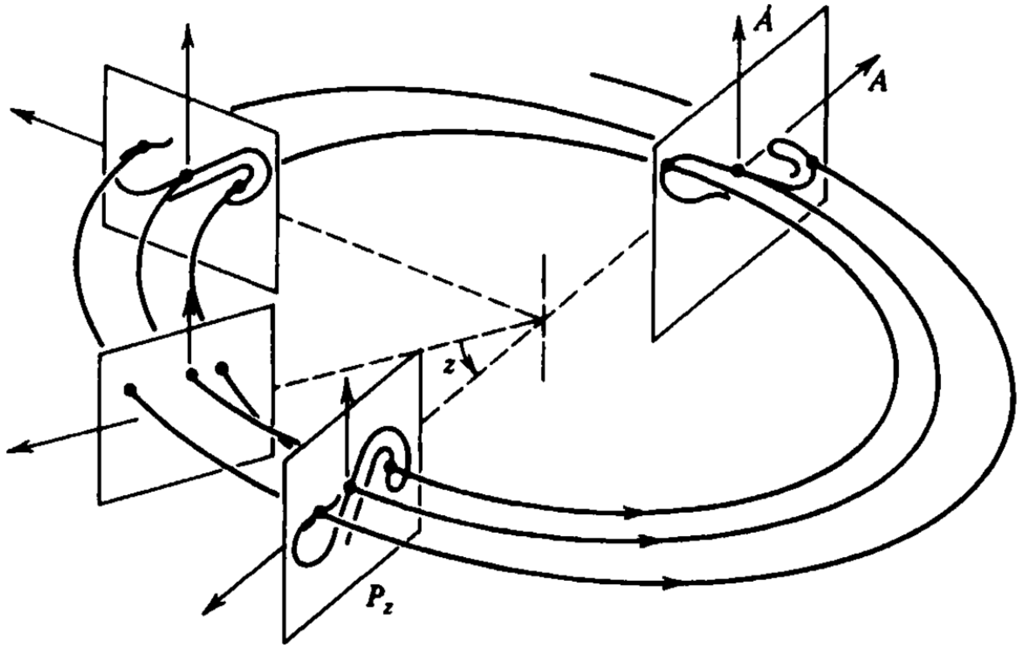
# Mapas de Poincaré

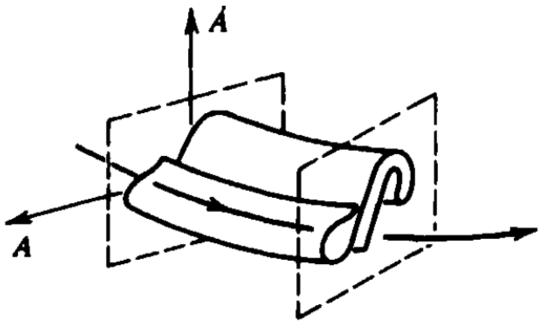


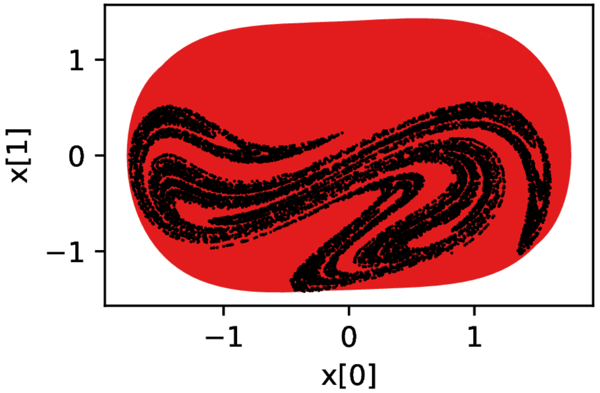



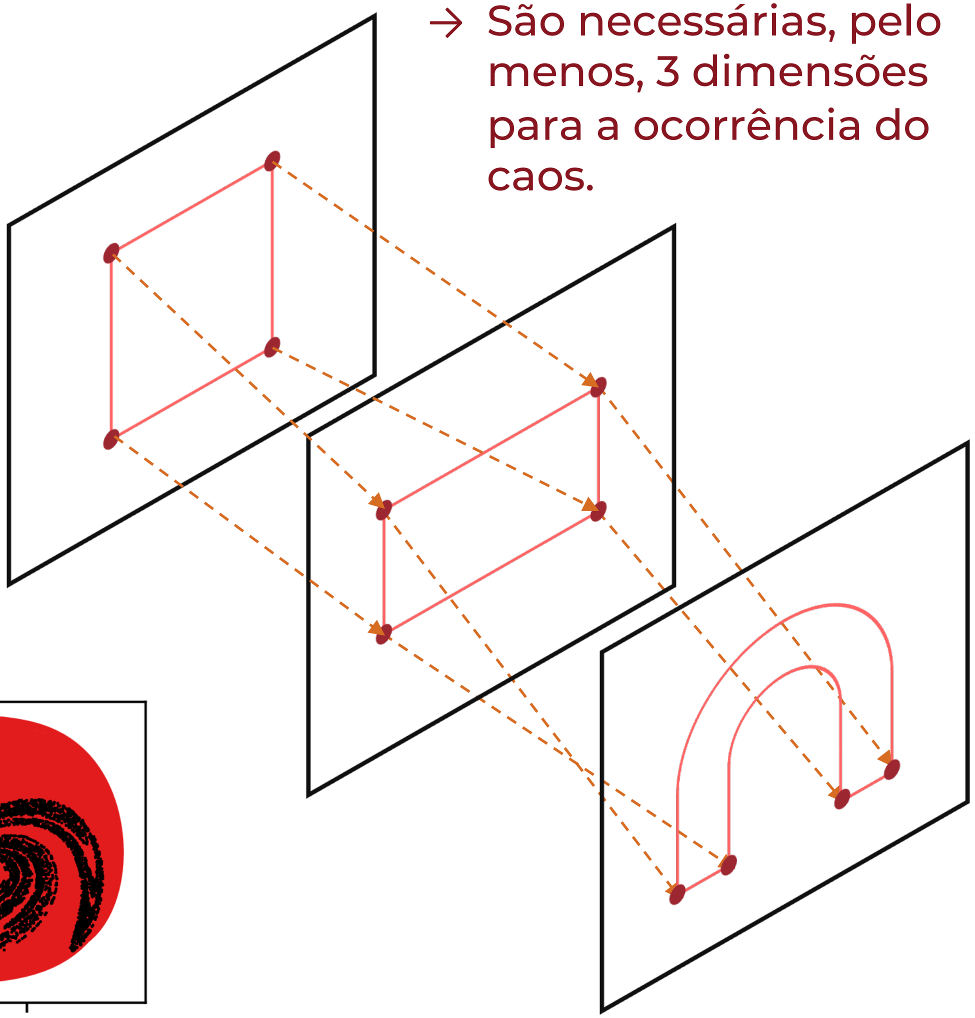



# Mapas de Poincaré



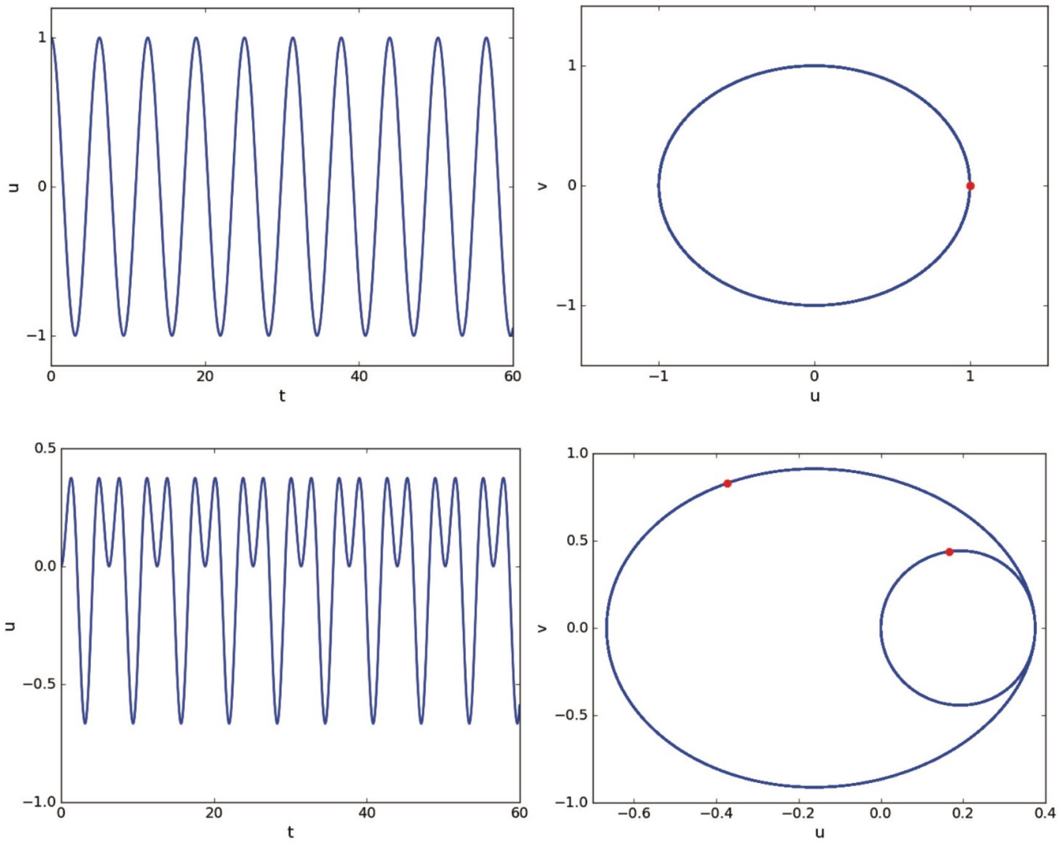



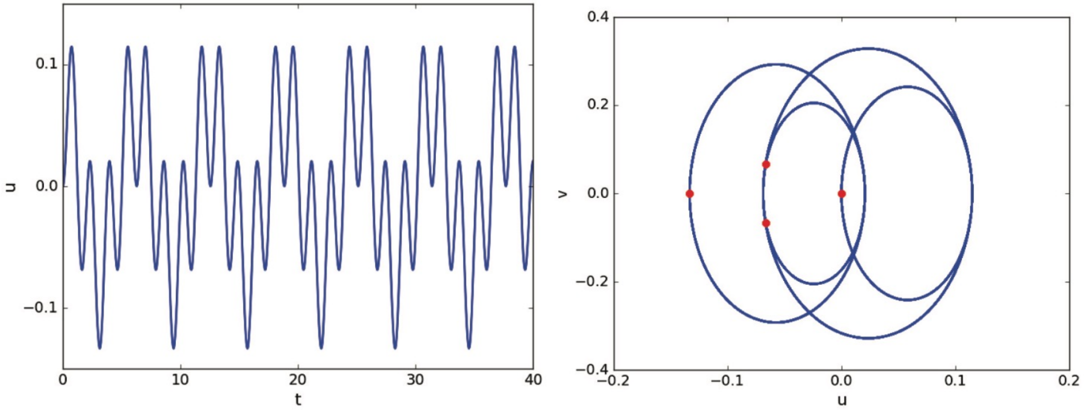



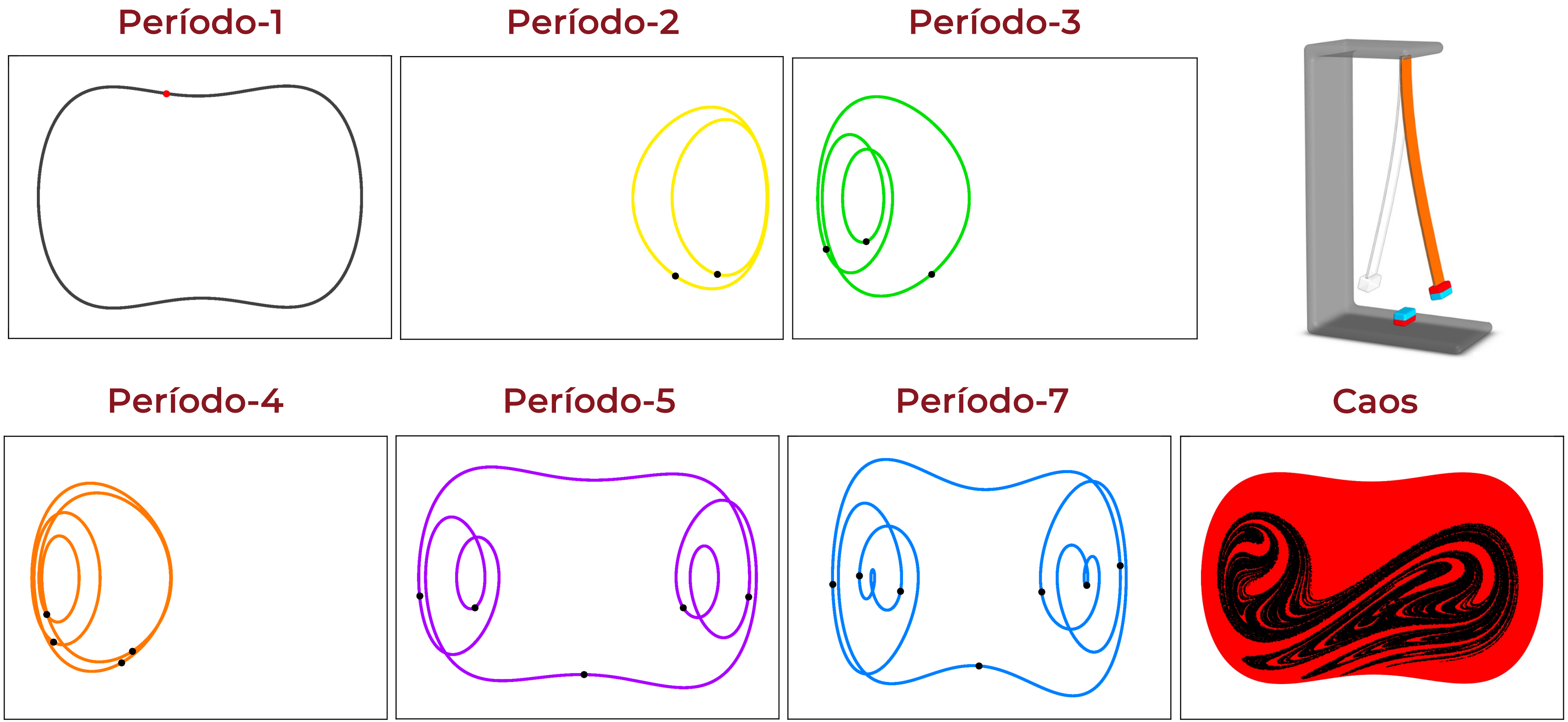
→ São necessárias, pelo menos, 3 dimensões para a ocorrência do caos.

# Mapas de Poincaré

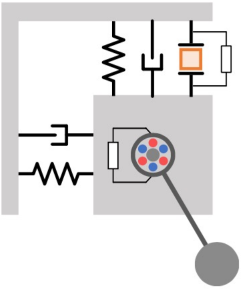


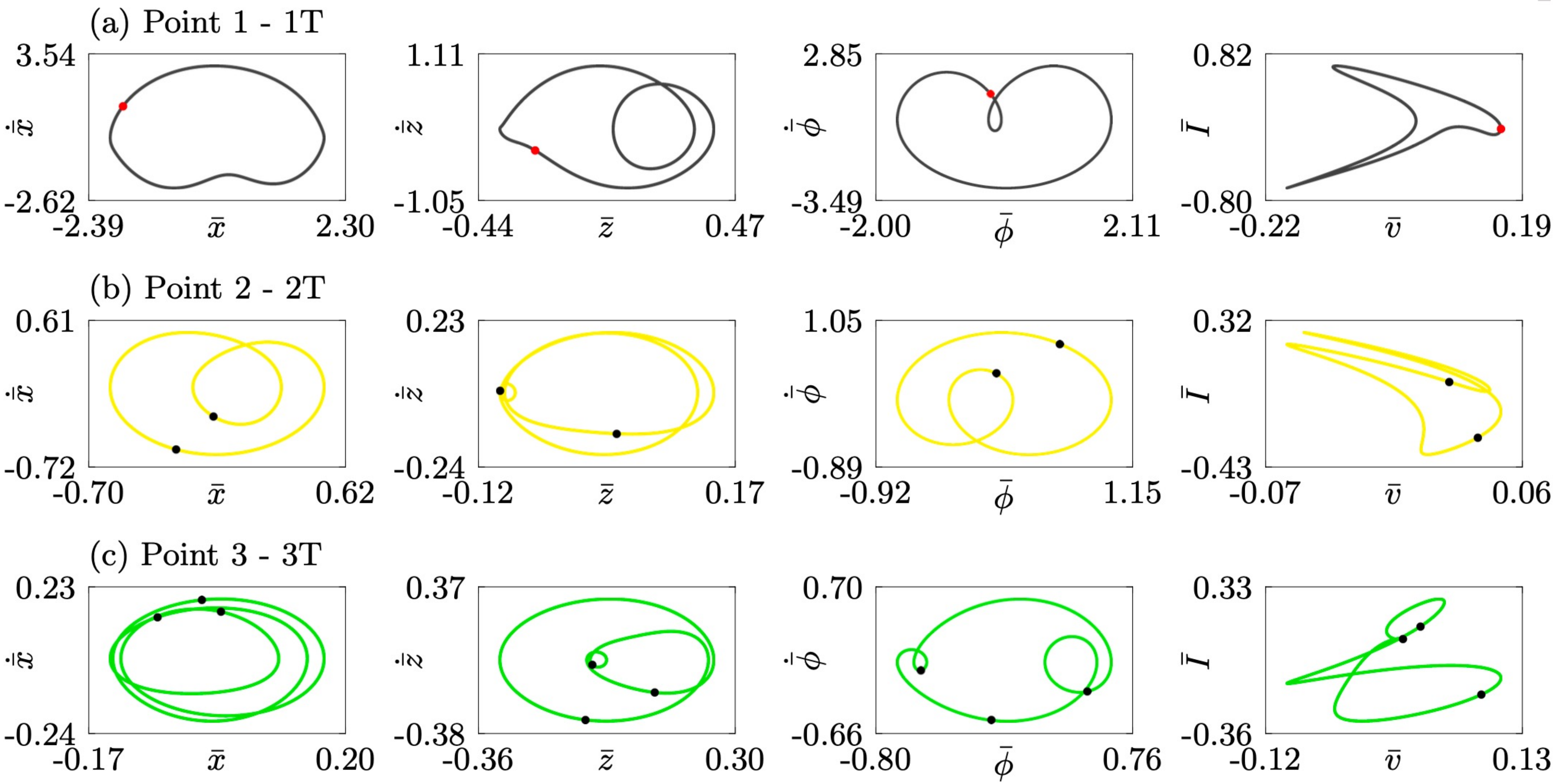



# Mapas de Poincaré

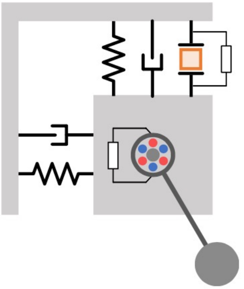


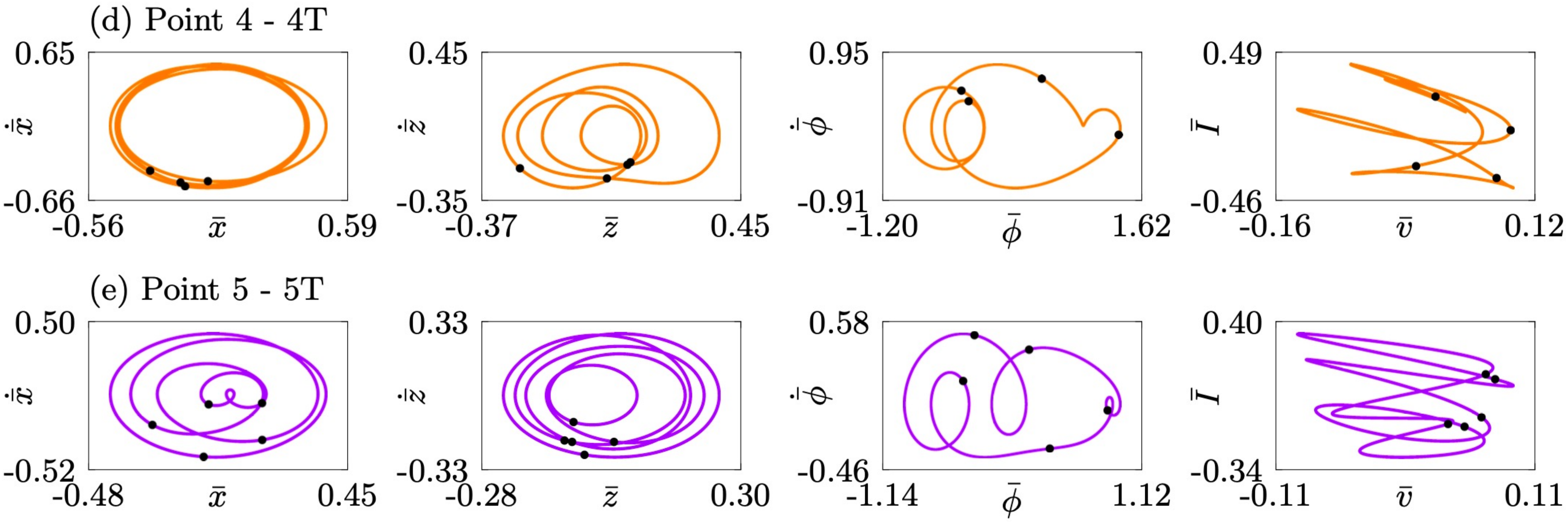
# Mapas de Poincaré



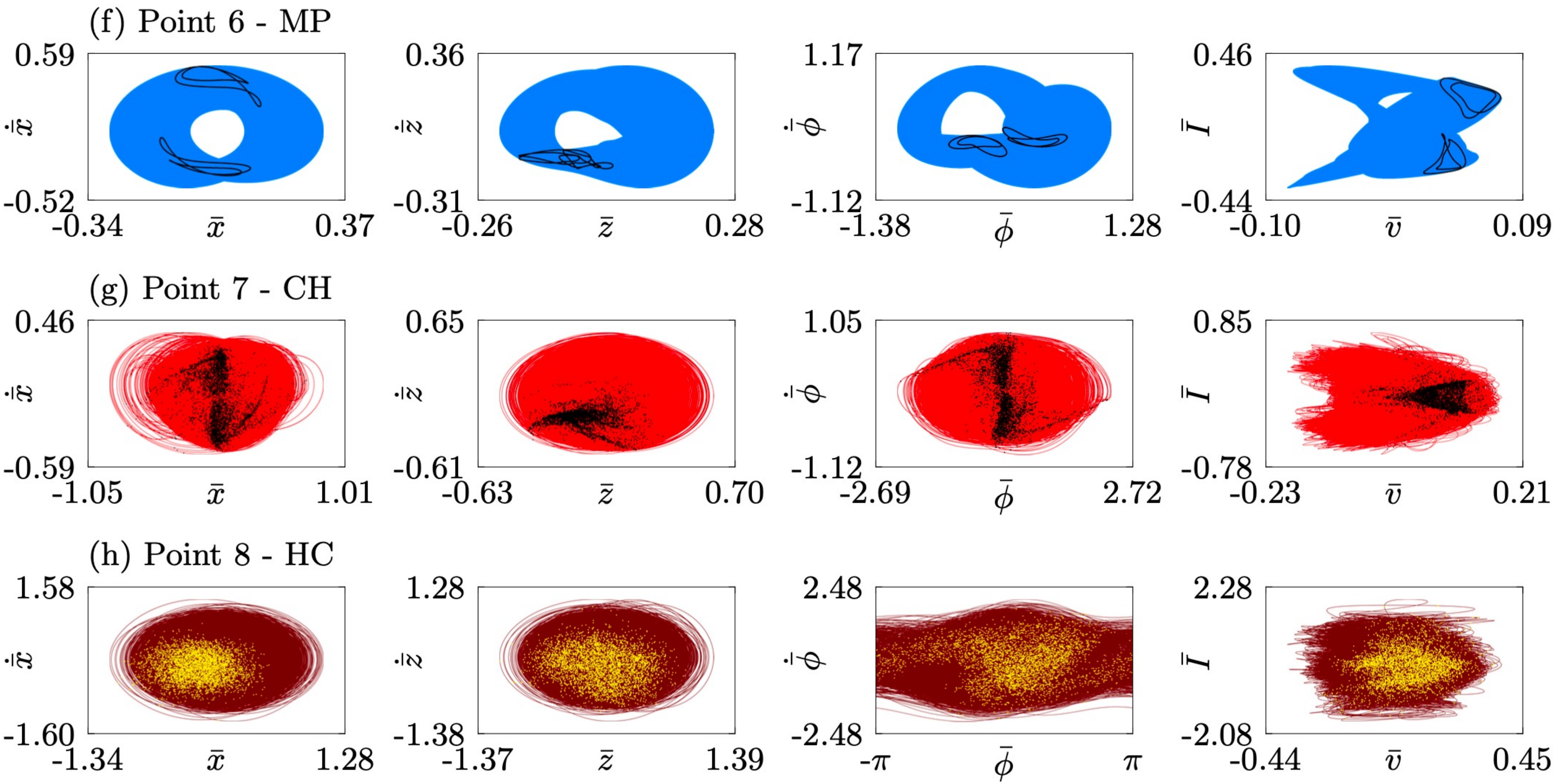


# Mapas de Poincaré

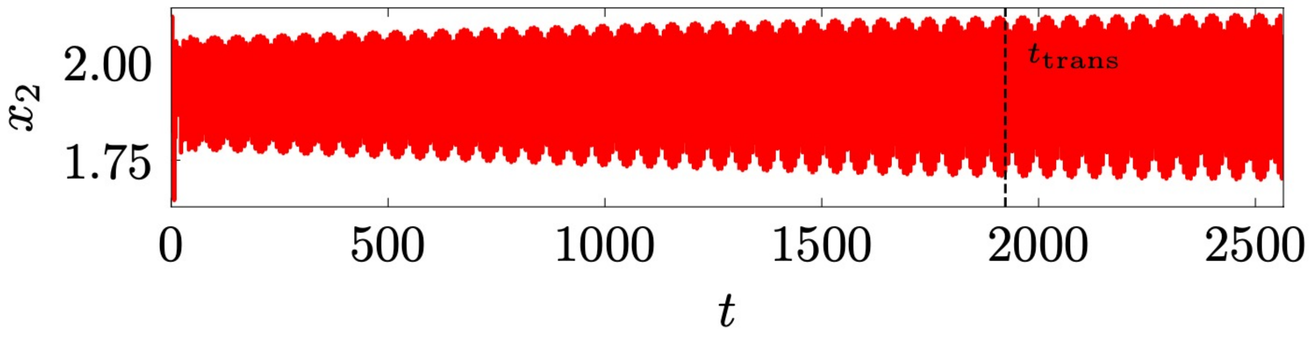


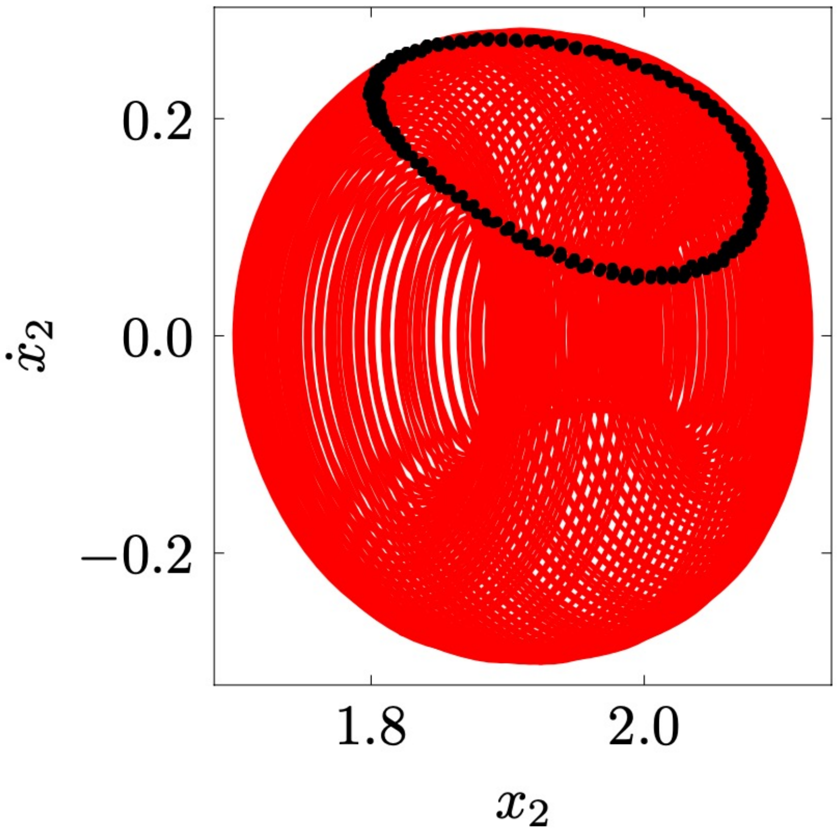


# Mapas de Poincaré

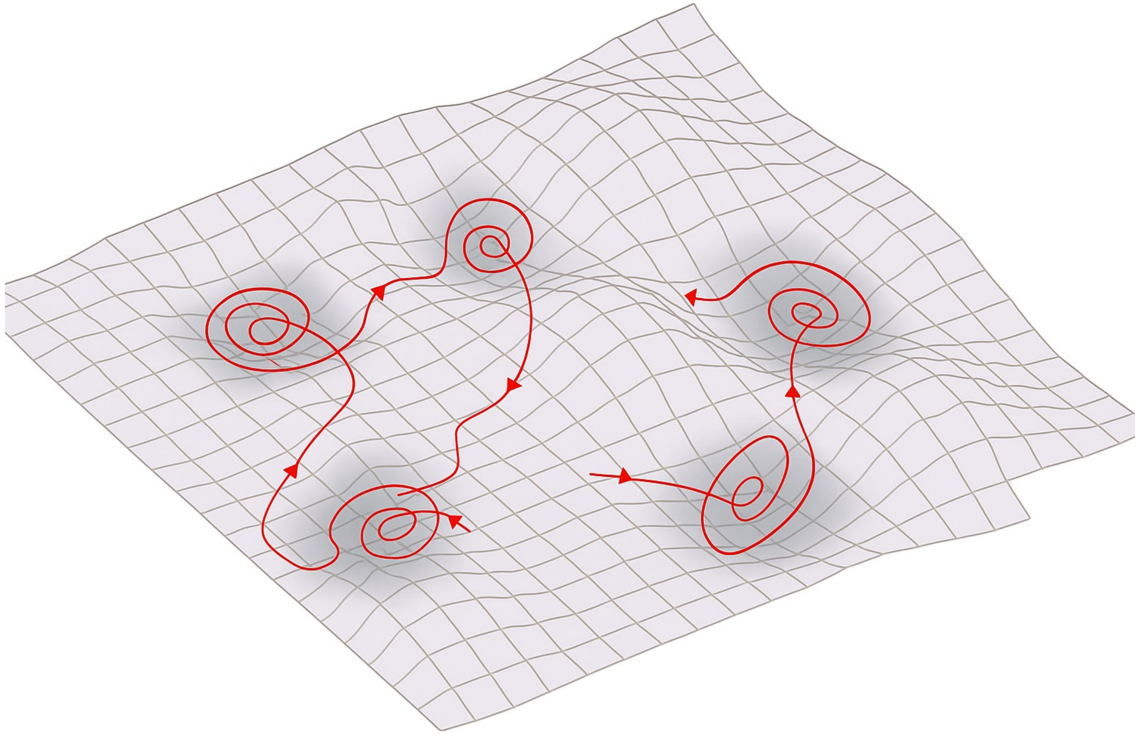


# Movimento Quasiperiódico

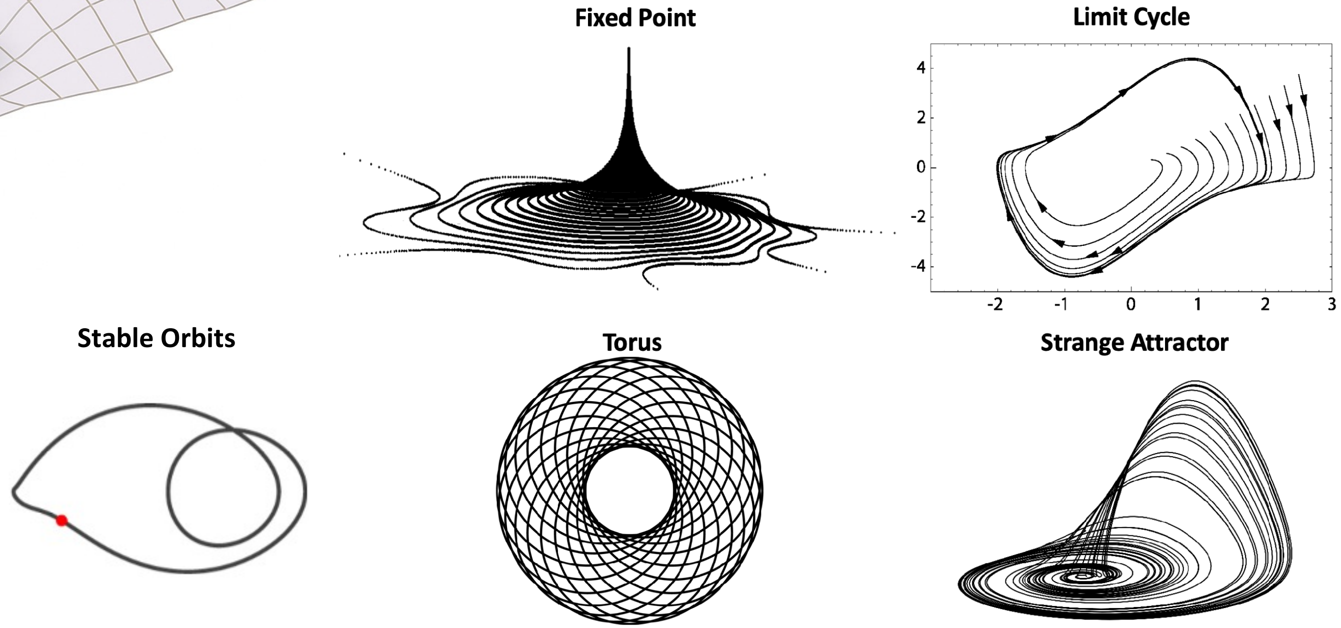


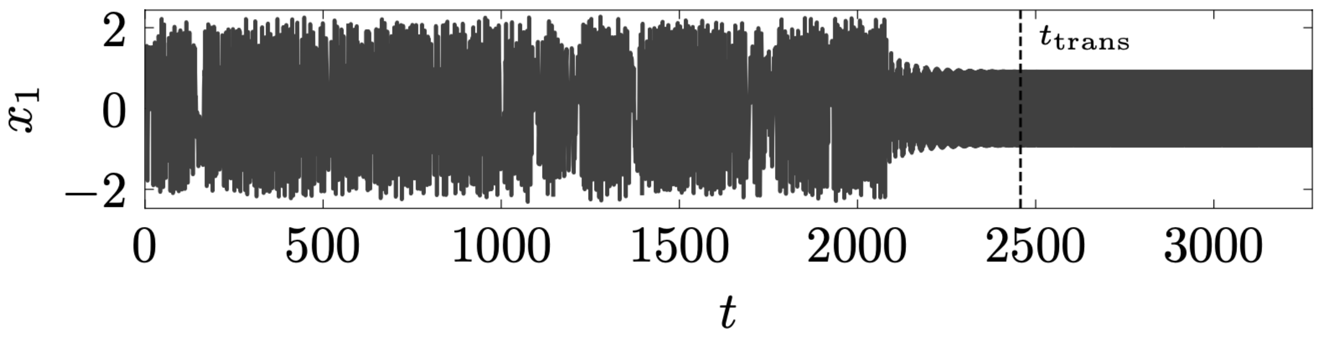
# Atratores

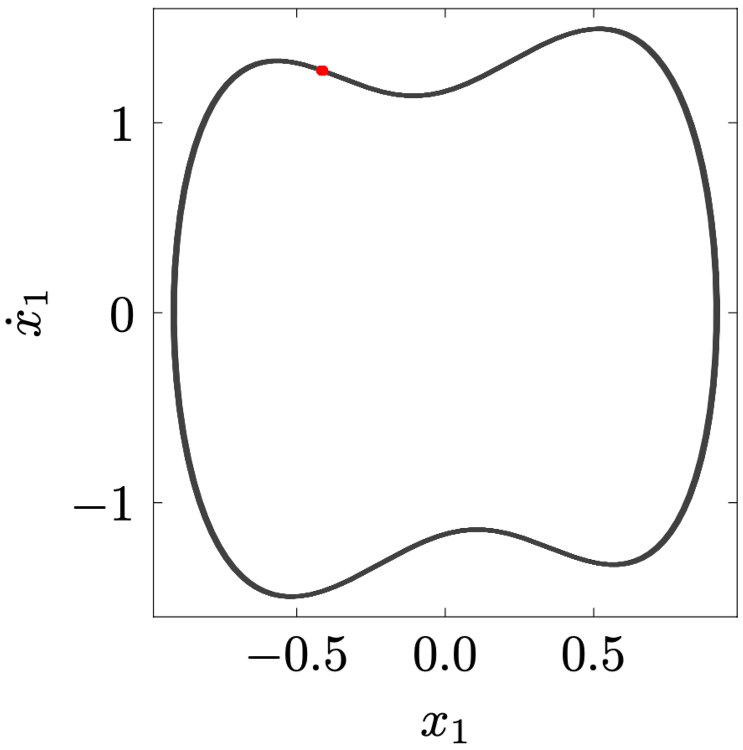


- → Um atrator pode ser visto como uma região preferencial no espaço de fase que atrai os estados de um sistema dinâmico.
- → Região na qual o sistema tende a evoluir para uma grande variedade de condições iniciais (Comportamento assintótico).

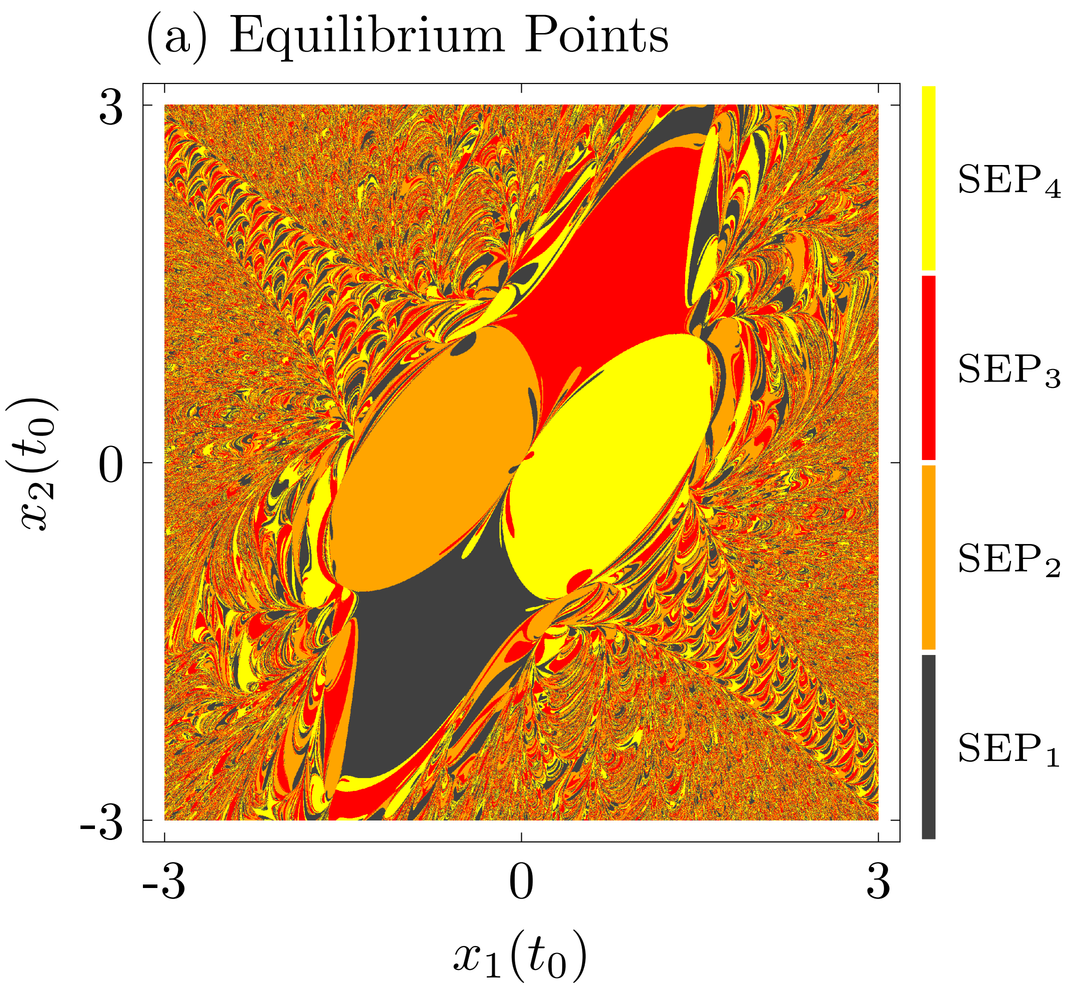


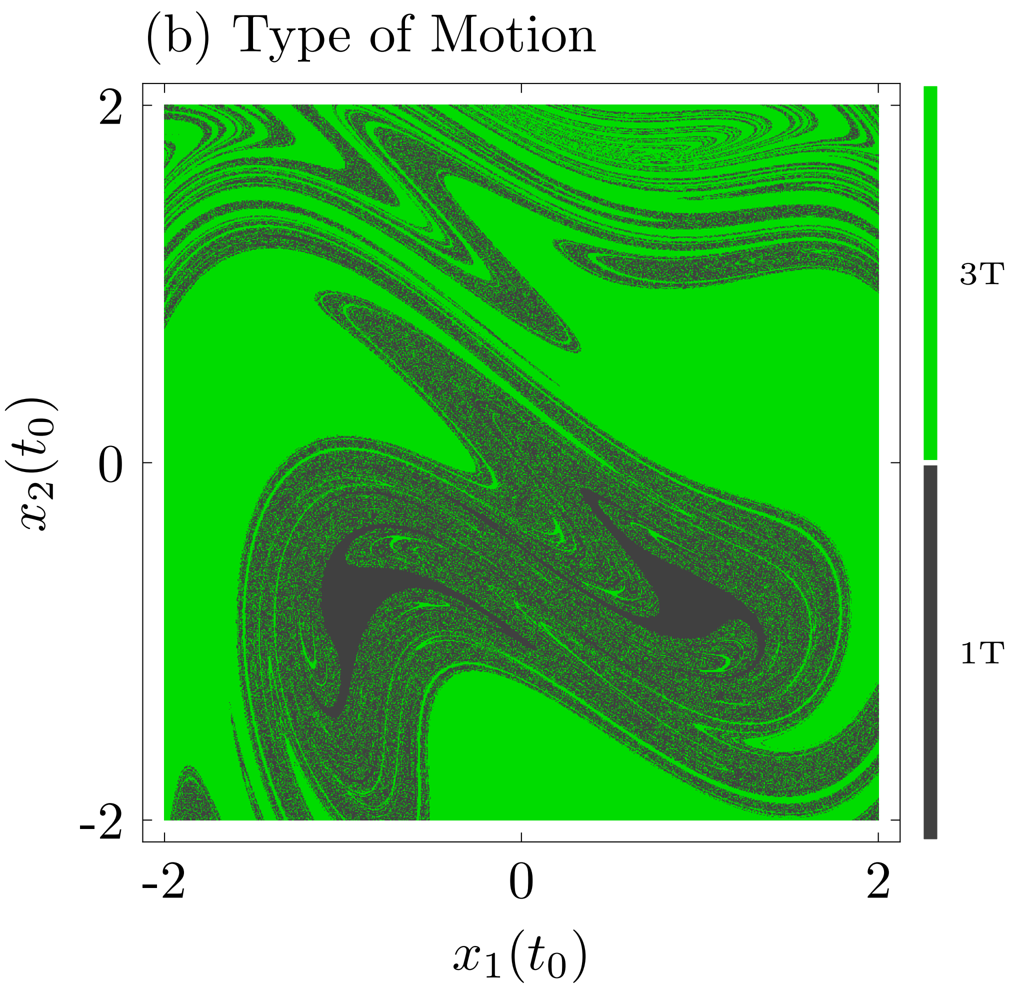
# Caos Transiente



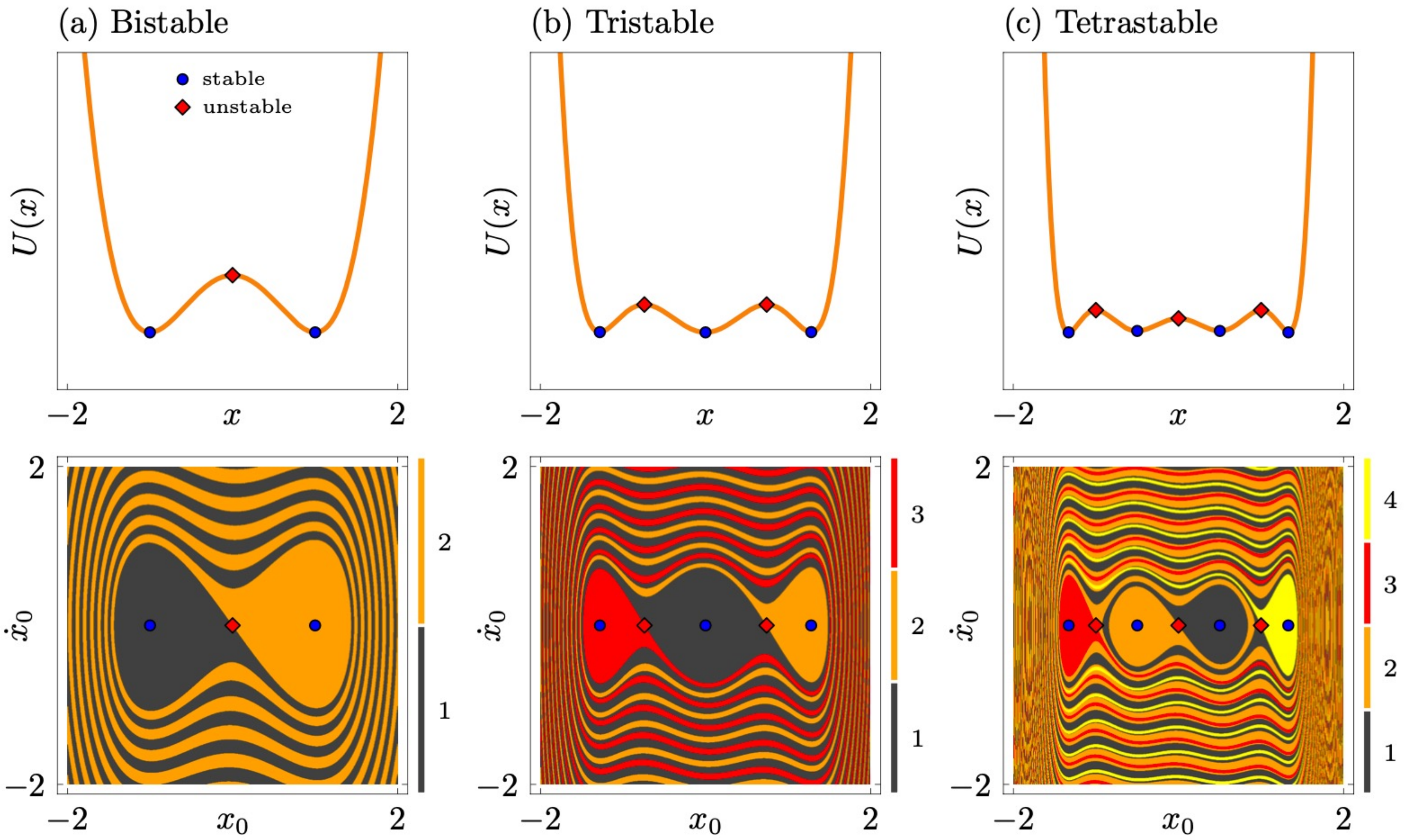



# Bacia de Atração

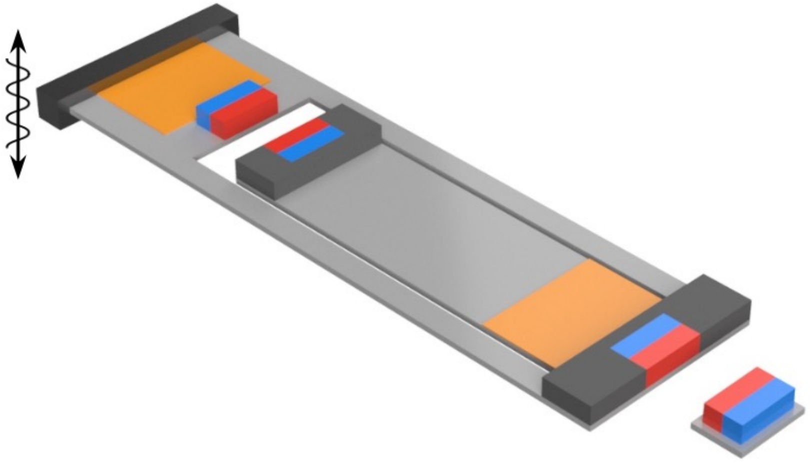


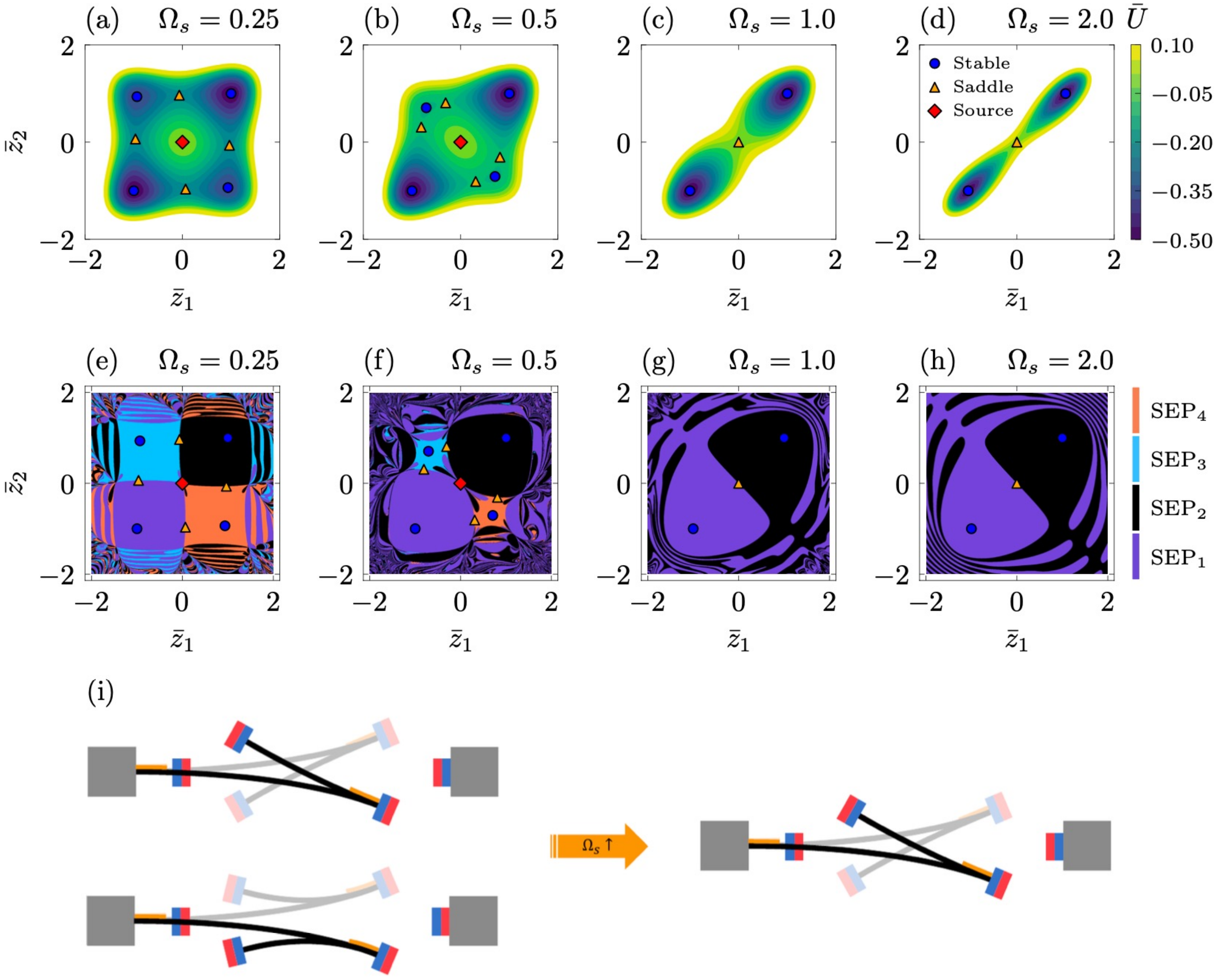



# Bacia de Atração

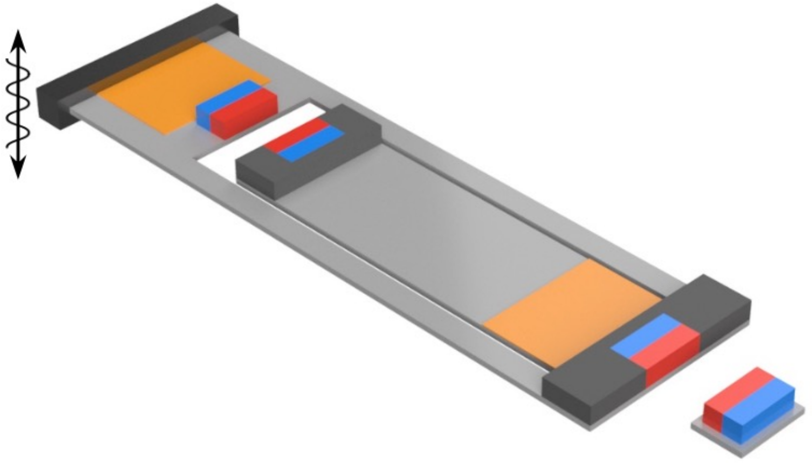


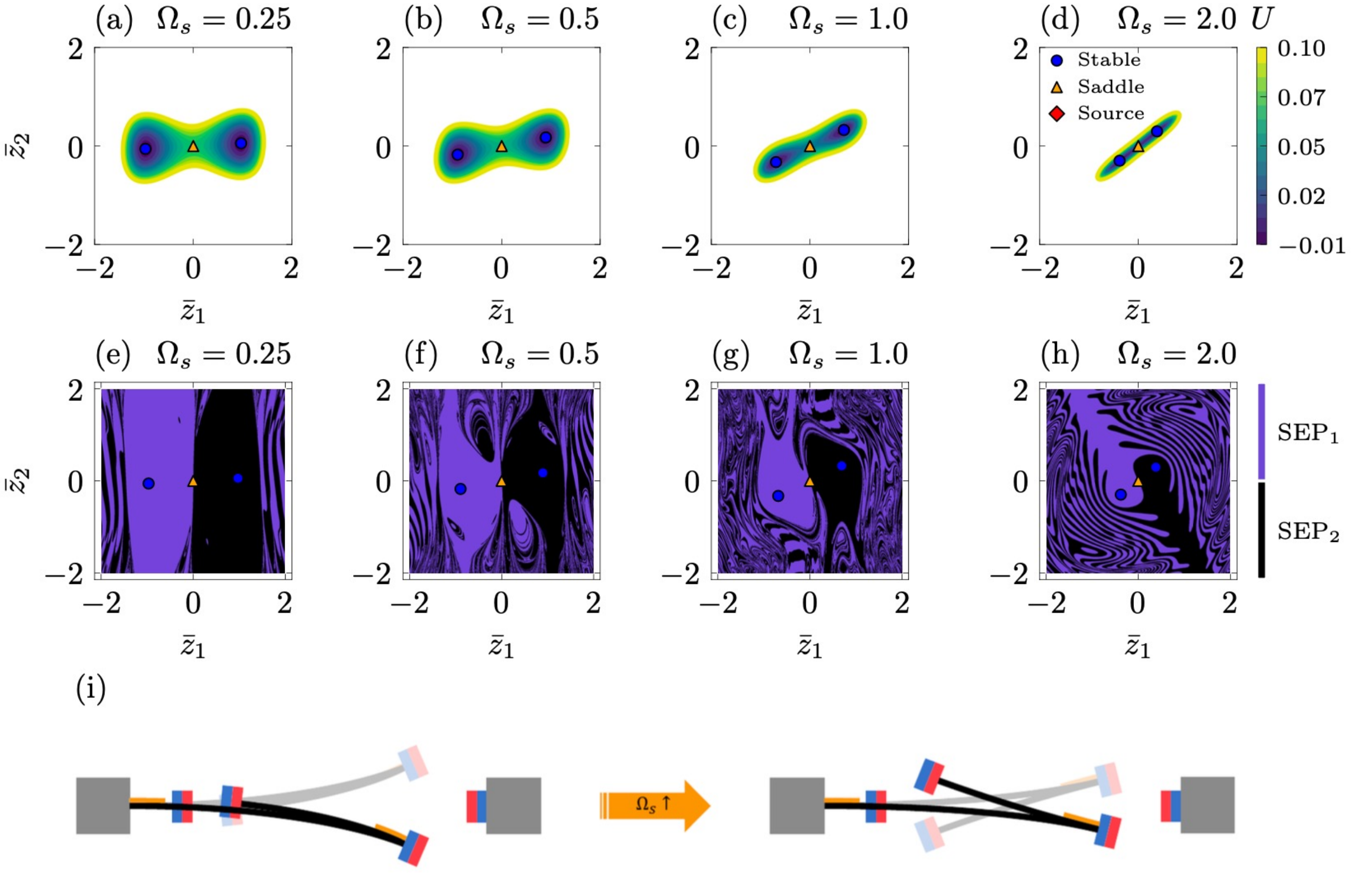
# Análise Global de um Sistema



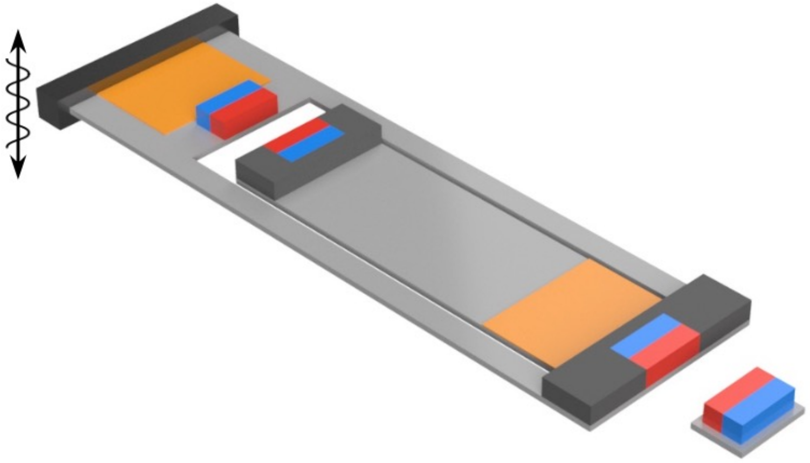


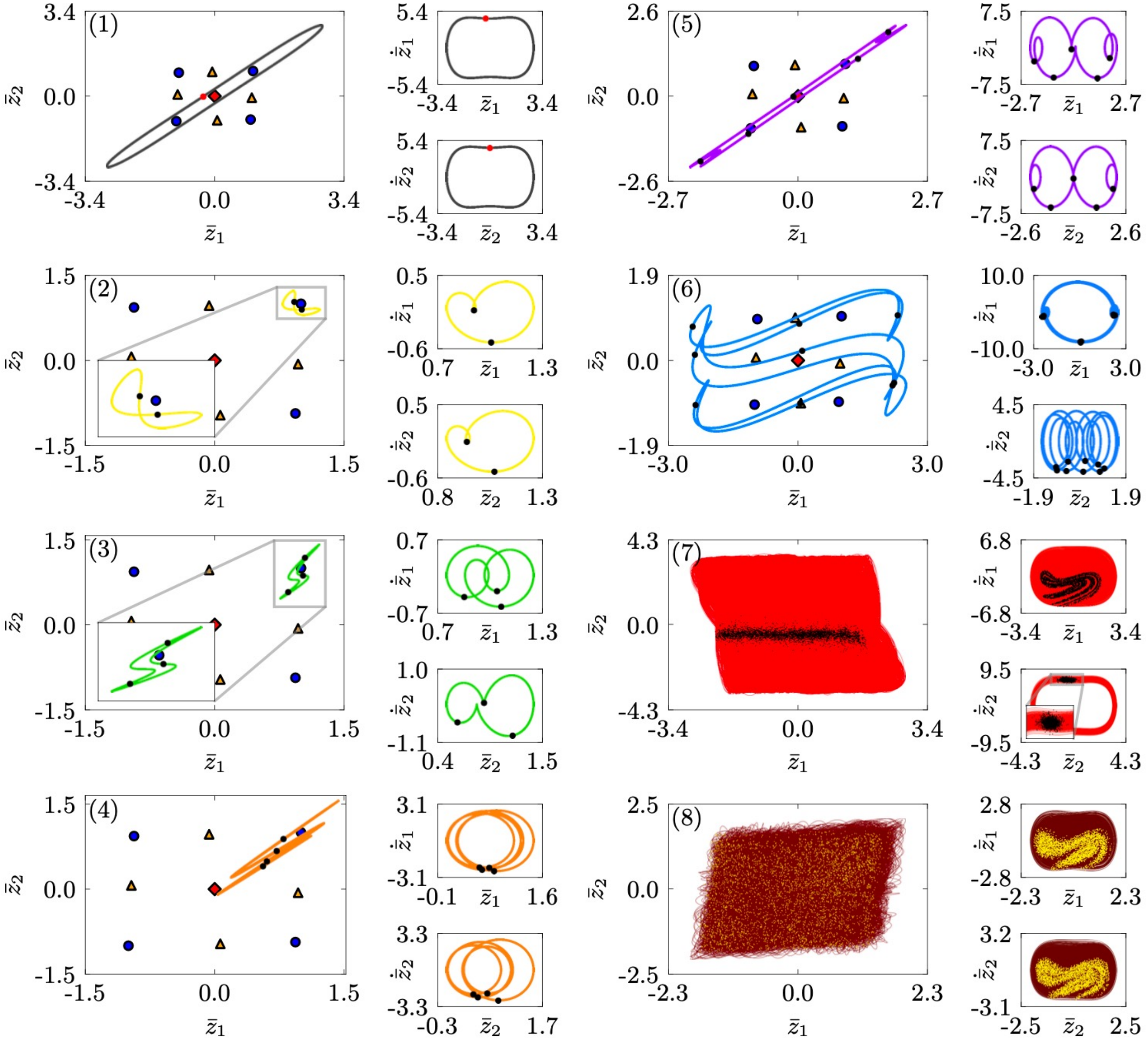
# Análise Global de um Sistema



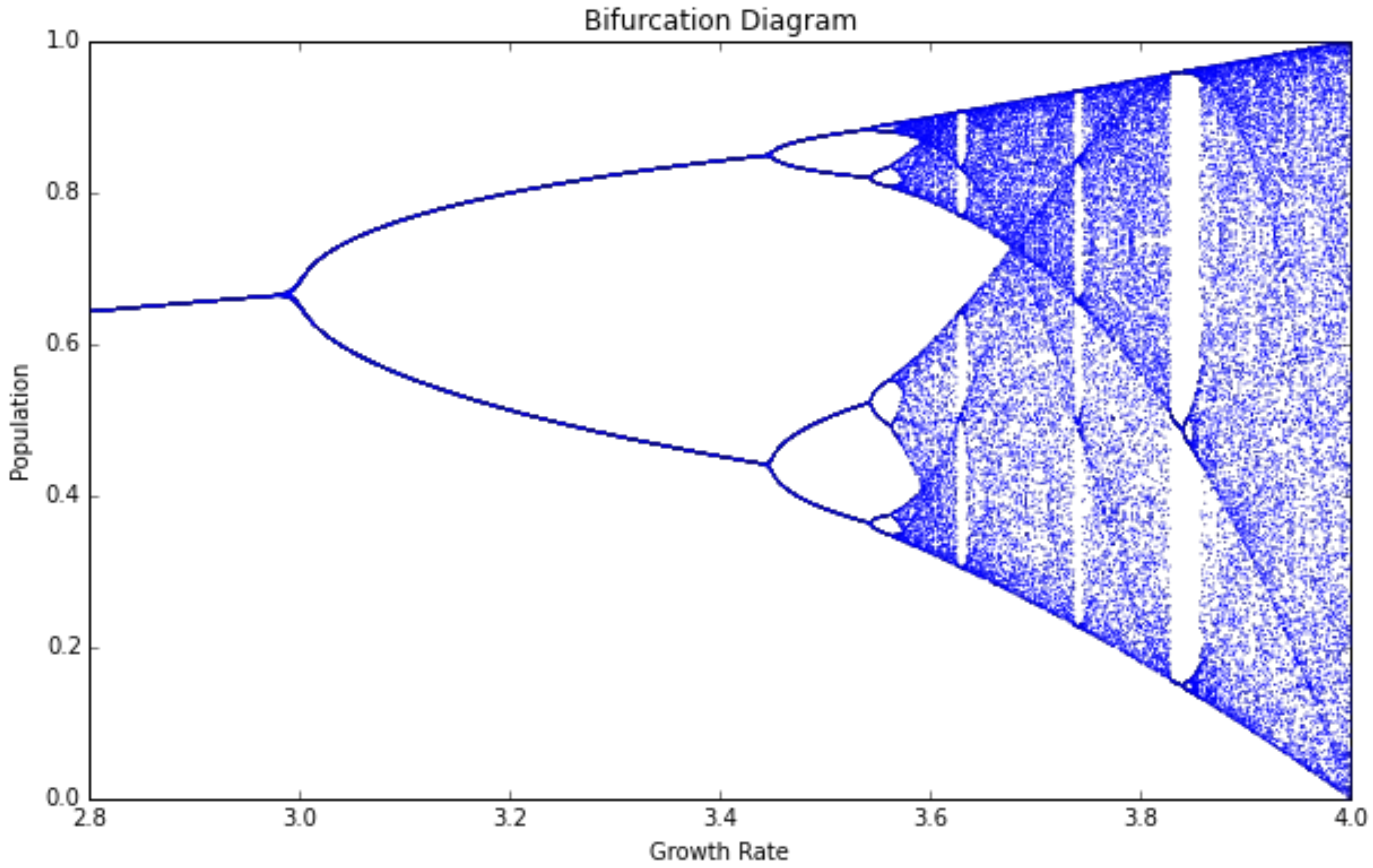


# Análise Global de um Sistema

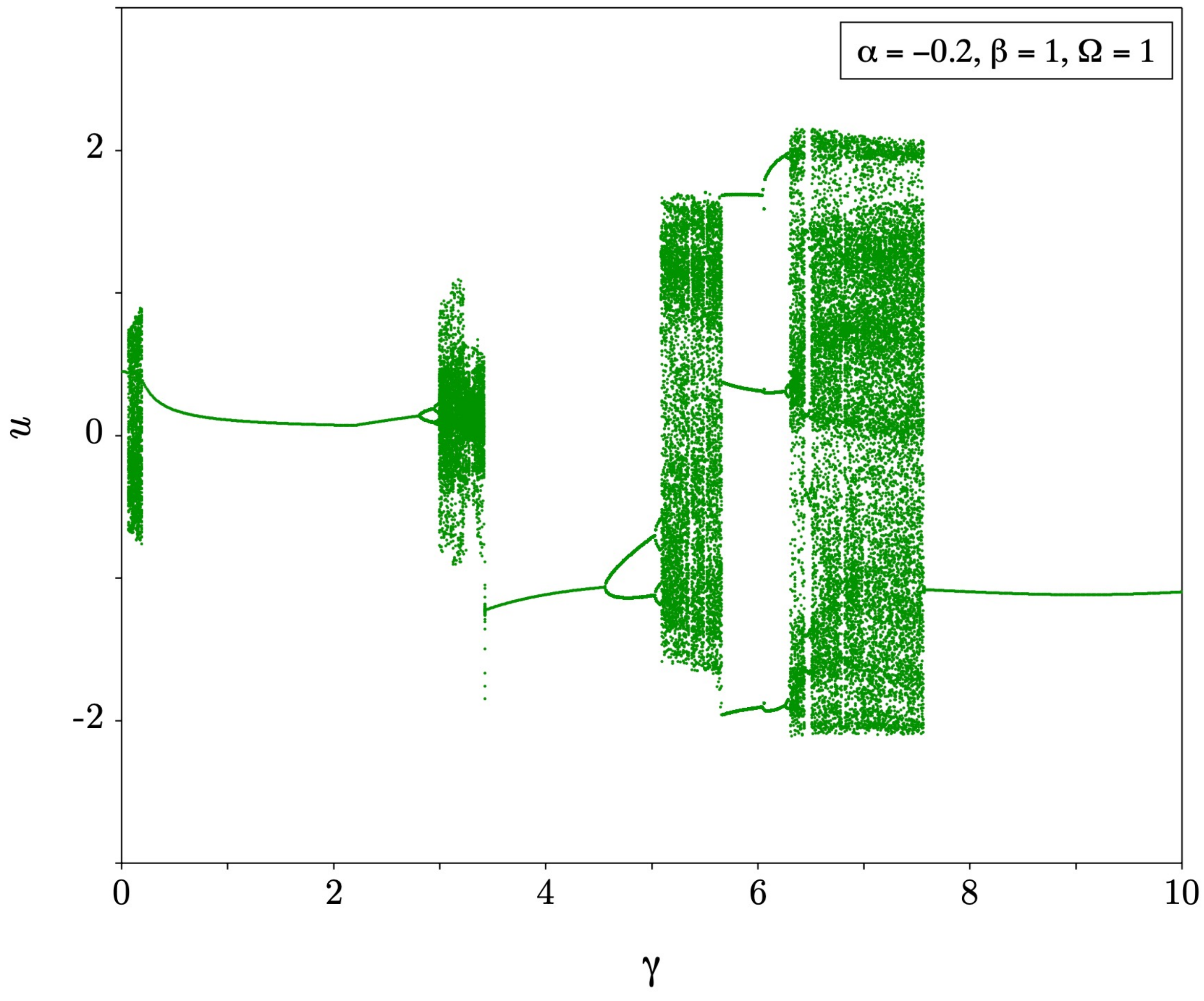




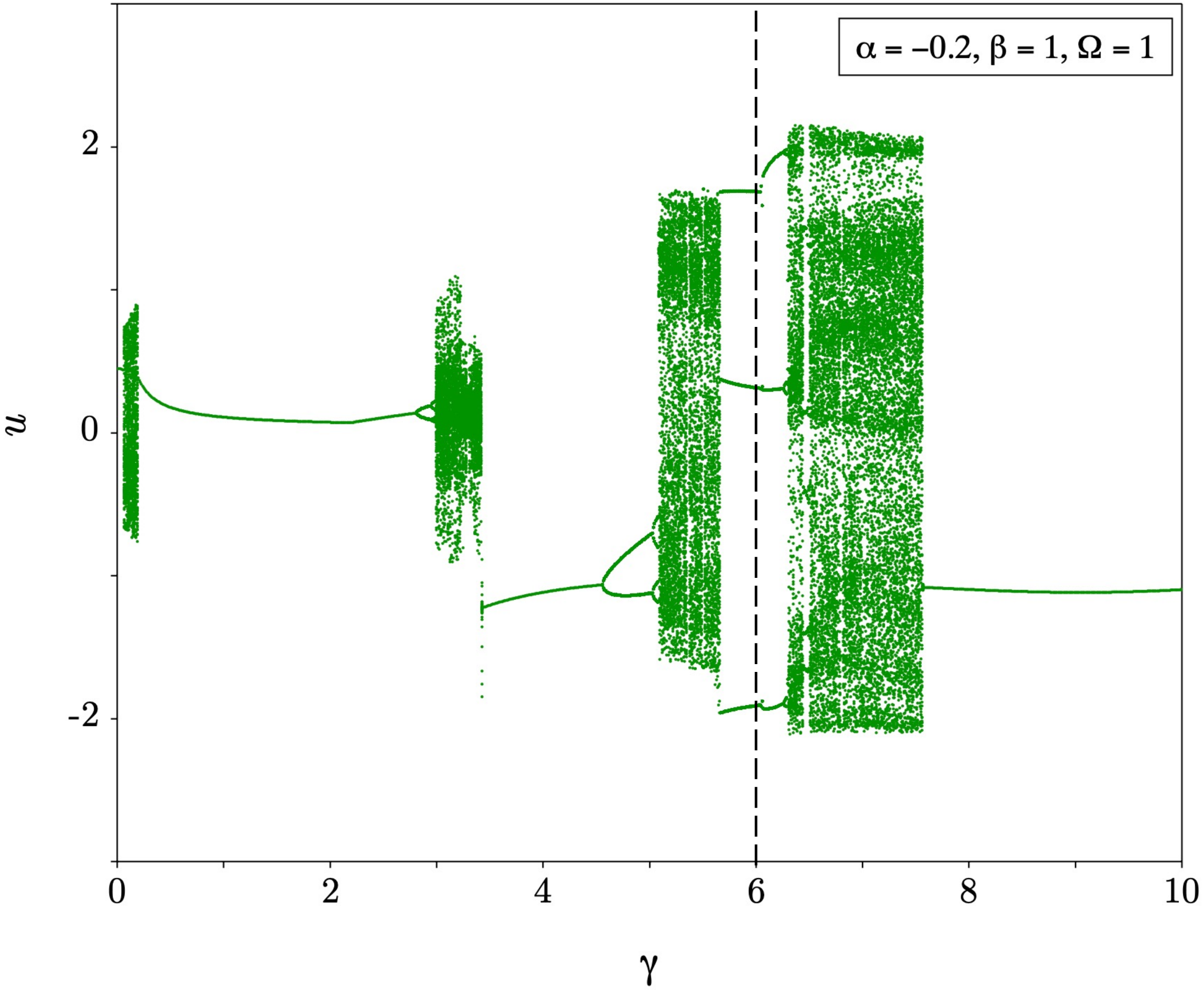
# Diagramas de Bifurcação

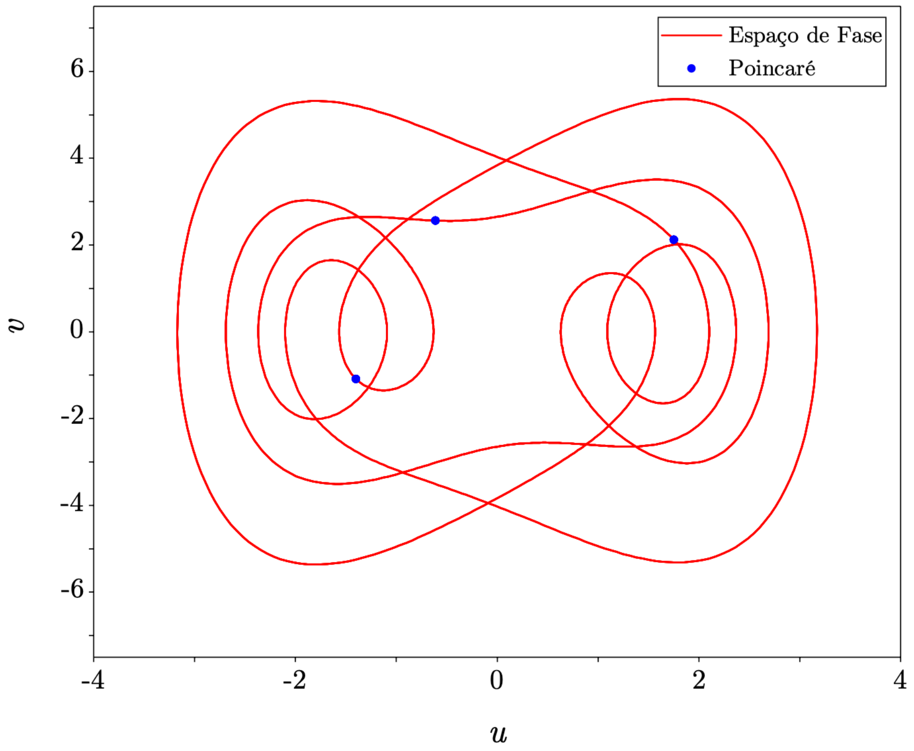


# Diagramas de Bifurcação

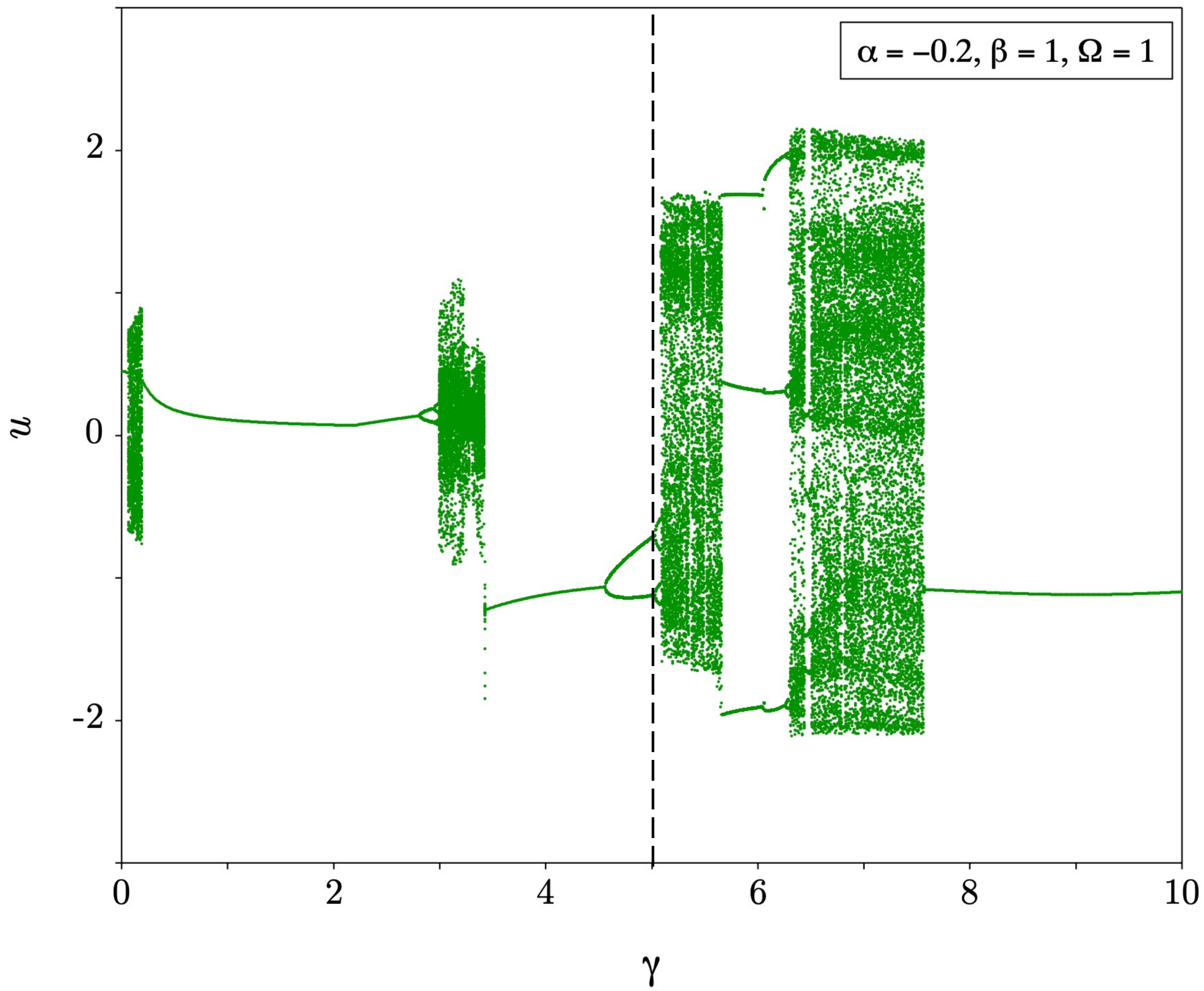


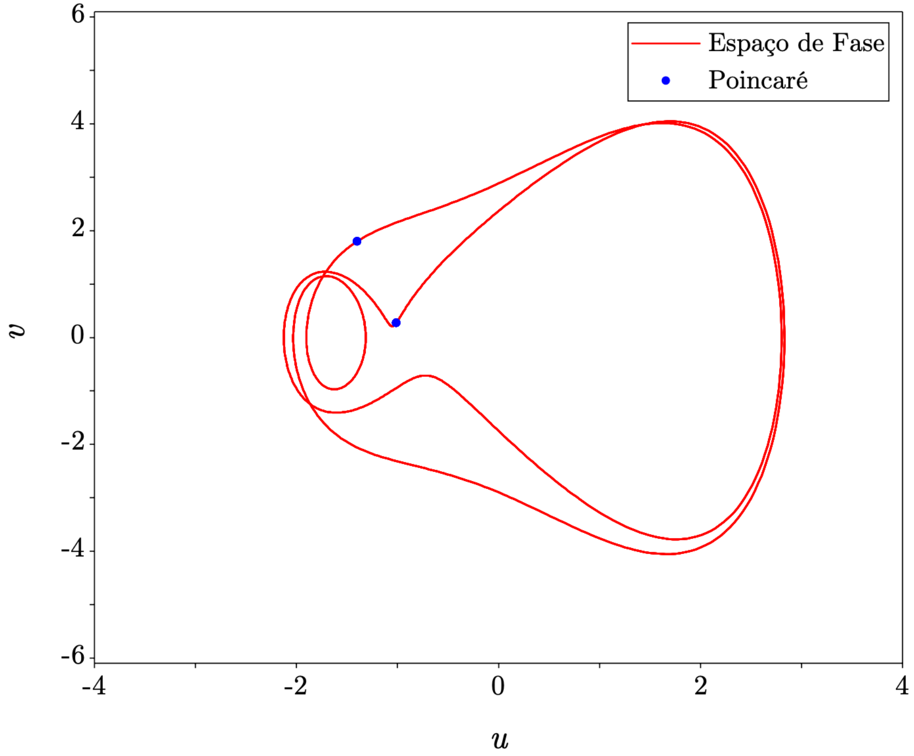
# Diagramas de Bifurcação



# Diagramas de Bifurcação



# Diagramas de Bifurcação